A STOCHASTIC MODEL FOR FATIGUE AND OPTIMUM DESIGN AND MAINTENANCE METHODOLOGIES

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A STOCHASTIC MODEL FOR FATIGUE AND OPTIMUM DESIGN AND MAINTENANCE METHODOLOGIES

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DEDICATION

I dedicate this thesis to my wife, parents, brothers and sisters, without whose loving inspiration and kind support, I could not have done justice to the research.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

NOMENCLATURE

a	half crack length
a_c	half critical crack length
b	half stringer spacing
C_I	cost of one inspection
C_R	tip stress reduction factor
C_s	cost of structure
C_T	total cost function
$D(k)$	NDI's detection probability at a crack length of $k(\Delta\ell)$
$E[\cdot]$	expectation of $[\cdot]$
$E_{\cdot}$	event of $(\cdot)$
h	breadth of the plate
K	stress intensity factor
k	crack length in units of $(\Delta\ell)$
k_c	critical crack length in units of $(\Delta\ell)$
K_{1c}	plane stress fracture toughness
k_r	repair threshold crack length in units of $(\Delta\ell)$
ΔK	range of stress intensity factors
L_{max}	peak load per unit width
L_{min}	minimum load per unit width
ΔL	range of load
N	number of cycles
N_{max}	maximum number of units of $(\Delta\ell)$ corresponding to maximum crack length for normalization

N_{st}	number of stringers
$P(\cdot)$	probability that $(\cdot)$
P_j	probability of failure under 'j' inspections and repairs
r	load ratio$=L_{min}/L_{max}$
R	reliability
R_s	static reliability
R_f	fatigue reliability
s	random factor of safety-ratio of σ_R to σ_L
T_d	design life of structure
T_o	time interval between the periodic inspections
t	thickness of plate
$Var[\cdot]$	variance of $[\cdot]$
w	total width of plate
W_{st}	weight of one stringer
W	total weight function
Z	normalization sum
α	shape parameter of Weibull distribution
β	scale parameter of Weibull distribution
λ	density of material of panel
σ_{Rs}	residual strength
σ_L	applied stress
$\sigma_{\cdot}^2$	variance of $\cdot$
$\mu_{\cdot}$	mean of $\cdot$
ν_k	mean crack growth rate in units of $(\Delta \ell)$ at a crack length of $k\Delta\ell$

SUMMARY

In this thesis a methodology is developed for cost-effective fatigue design of structures subject to random fatigue loading. A stochastic model for fatigue crack propagation under random loading has been discussed. Fracture mechanics is then used to estimate the parameters of the model and the residual strength of structures with cracks. The stochastic model and residual strength variations have been used to develop procedures for estimating the probability of failure and its changes with inspection frequency. This information on reliability is then used to construct an objective function in terms of either a total weight function or cost function. A procedure for selecting the design variables, subject to constraints, by optimizing the objective function has been illustrated by examples. In particular, optimum fail-safe design of stiffened panel has been discussed.

The fail-safe design procedure, depends on the types of damage that are possible in an aircraft structure. In general, damages are due to fatigue, stress corrosion and foreign objects. In this thesis, only fatigue damages have been considered. Fatigue damages are characterized by fatigue cracks. Catastrophic damage can take place depending on the length of the crack and the stresses in the material. For

a given material, fail-safety against fatigue damage can be provided by increasing the initial margin of safety and testing or by a number of inspections and repairs during the service life of the structure. Increase of initial margin of safety increases the weight of the structure and possible loss of performance. Decreasing the initial margin of safety and increasing the inspection frequency also has a similar beneficial effect on fail-safety with reduced weight of the structure. However, the cost due to inspection, increase. Therefore, there is a need to optimize the cost or equivalent weight function to obtain the appropriate design variables such as the initial margin of safety, inspection frequency, etc. These optimizations are subject to the restraint of prescribed reliability bounds. Therefore, in order to meet these requirements, a simple procedure for reliability-based, cost-effective, fail-safe design for fatigue has been discussed in the thesis.

Also, a methodology for the reliability analysis of a reusable solid rocket motor case has been discussed in this thesis. The analysis is based on probabilistic fracture mechanics and probability distribution for initial flaw sizes. The developed reliability analysis can be used select the structural design variables of the solid rocket motor case on the basis of minimum expected cost and specified reliability bounds during the projected design life of the case. Effects of fracture control plans such as non-destructive inspection and the material erosion between missions can also be considered in the developed procedure for selection of design variables. The reliability-based procedure that has been discussed in this thesis can easily be modified to consider other similar structures of reusable space vehicle systems with different fracture control plans.

CHAPTER I

INTRODUCTION

Fatigue, in general, is defined as the progressive failure of a component under repeated, cyclic or fluctuating loads with the presence of tensile loads [1]. The problem of fatigue is inherent to most of the aircraft structures and fatigue damage is one of the prime reasons for the reliability impairment [2]. The damage due to fatigue manifests itself in the form of growing cracks in critical locations of the structure undergoing fatigue. Failure of the structure occurs when the fatigue cracks grow beyond the critical length. The critical length is that length at which the stress intensity factor for a given load equals the fracture toughness of the material of the structure [3].

The fatigue damage can be identified and repaired if timely inspections are performed and the cracks are repaired during systematic maintenance schedules [4]. The selection of the maintenance schedule depends not only on the fatigue behavior of the structure but the reliability level demanded. A schedule with more number of inspections for fatigue cracks during the lapse of the design life imparts a higher reliability and vice versa. While a highly fatigue prone structure necessitates a closely knit inspection schedule, a structure in light fatigue environment does not need as many inspections and repairs. All of these factors can be judiciously translated into monetary values. A mathematical model can be constructed involving the above factors for determining the optimum number of inspections for fatigue damage.

Since "prevention is always better than cure," the problem of fatigue damage can be tackled from a more fundamental point of view. That is, to include the fatigue damage considerations in the design of the structure. The current fatigue design of aircraft structures is predominantly deterministic. However, some effort has been put forward towards probabilistic fatigue design methodology by a few authors [5, 6-10]. This effort is restricted to the use of probabilistic models for the crack initiation time or failure time, to express the probability of failure, and to use the probability of failure in optimizing the structure. The disadvantage of the procedure is that the probabilistic description of fatigue phenomenon with a single random variable is not adequate for the following reason. The mechanism of fatigue damage involves the changing probability distribution with increasing number of cycles and is generally in three stages from a macroscopic point of view. Crack growth time from microsize to detectable size can be lumped into the first stage of the following three stages comprising the fatigue process [9]

(1) crack initiation

(2) crack growth, and

(3) failure.

The inadequacy of the present probabilistic methods that use a single random variable is that they do not encompass all of the above stages of the fatigue phenomenon [96]. The development of a methodology for the reliability based optimum design of aircraft structures necessitates a model that encompasses all stages of progressive damage from micro-size to cracks of critical lengths. This is in turn helpful for the proper prescription of inspection and maintenance schedule also. This can be accomplished by developing a stochastic model for fatigue that describes the changing probability distribution of cracks with increasing flight hours or cycles. The objective of this thesis is to develop such a model and demonstrate its application to optimum structural design problems.

Numbers in parentheses for this chapter belong to the supplemental bibliography.

CHAPTER II

ANALYSIS OF INSPECTION DATA FROM A FLEET

In the previous chapter, the necessity for a stochastic model for fatigue is explained based on the mechanics of progressive fatigue damage, i.e. the changing probability distribution of crack length with increasing number of cycles. Before proceeding with the development of a stochastic model, investigations have been conducted to see if the probability distributions of fatigue cracks indeed change with increasing number of flight hours in an aircraft fleet.

This analysis consisted of analyzing the fatigue failure data from a specific fleet of aircraft [102]. Fatigue failure data of the center box wing of this fleet of aircraft is analyzed. There are 94 critical locations chosen on the wing surface which are susceptible to fatigue. Fatigue crack information at these critical locations is recorded at selected inspection times. Some of the relevant items comprising the data are as follows:

(a) flight hours completed

(b) length of the crack

(c) location of the crack, and

(d) name of the base and command of operation.

In the preliminary analysis failure is "defined" as the mere appearance of a fatigue crack at an inspection time irrespective of the length of the crack. There is a great deal of scatter in these times to fatigue "failure." Many investigators have suggested that the time to fatigue failure fit a Weibull distribution [12,31,103]. Sometimes they have assumed the shape parameter and then estimated the scale parameter. However in the present research all the parameters of the proposed distribution are estimated from the data, by using maximum likelihood method. Also the probability distributions obtained have been checked by appropriate goodness-of-fit tests.

In the first analysis the failure criterion has been the first observation of a fatigue crack at the inspection time oblivious of its length. Next, the failure criterion is modified to be the time for the appearance of a crack of a given length. These times corresponding to the selected crack lengths are determined by regression analysis [114]. Then Weibull parameters are estimated for these times and once again checked for goodness-of-fit. This analysis indicates that the parameters of the distribution change for different selections of crack lengths in the failure criterion. This result of changing probability distributions with increasing crack lengths is indicative of the necessity of a stochastic process for crack length in time.

The details of the preliminary analysis are discussed in Appendix I. This appendix also describes a procedure for grouping fatigue data from different bases using the aircraft fatigue data. In particular, analysis of variance has been used to decide if the inspection data from different bases can be grouped and considered to have come from a single population.

Before attempting at the problem setting for this thesis a literature survey has been conducted. This is discussed in Appendix II.

CHAPTER III

PROBLEM SETTING AND THESIS OUTLINE

As discussed in the preceding chapters and as revealed by the literature survey of Appendix II, in the present state of art there is not an adequate technique neither for the probabilistic model of the entire stochastic process of fatigue nor for the reliability based fatigue design of aircraft structures that incorporates such a model. A way of describing the fatigue damage of an aircraft includes crack initiation to detectable size and crack growth to failure [96]. By virtue of the uncertainties involved in the external loading and the material properties the crack initiation and subsequent growth can be considered as random variables. This randomness is demonstrated by the observed crack lengths during service inspections as discussed in Appendix I. Thus, two random variables, namely the crack initiation time and the crack length need to characterize the fatigue process.

3-1. Fatigue Phenomenon as a Stochastic Process

In order to obtain probability distribution for crack length after initiation, a complete description of the stochastic process is necessary. Also, any development of such a stochastic process for fatigue phenomenon should be capable of explaining all the phases of fatigue process, i.e. initiation, growth and failure. In fact, the failure criterion derived from such a stochastic process would be more realistic than that employed in simple probabilistic models involving a single random variable. Such a failure criterion would be the exceedance of the critical stress intensity factor at the end of the intended design life time. Thus the first major contributory problem is to develop a stochastic model for a better probabilistic description of fatigue phenomenon. Such an accomplishment will be of immense help in the probabilistic, fail-safe, fatigue design of aircraft structures. Development of such a methodology for fail-safe design has also been investigated in this thesis and illustrated by example problems.

3-2. Design Problem: Description of Geometry

The first example is that of probabilistic fracture-critical design of stiffened sheet structure. The majority of aircraft structures are made up of sheet structures stiffened by intermittent stringers. Because of the randomly cyclic nature of the inplane loads acting on these sheet structures, the resulting fatigue cracks exhibit a great deal of scatter. Figure 3-1 illustrates a stiffened panel which is a substructure of a larger aircraft structure. The panel is of width 'w' and thickness 't'. The stiffeners are located on the top surface of the panel at a uniform spacing of '2b'. The external inplane loading

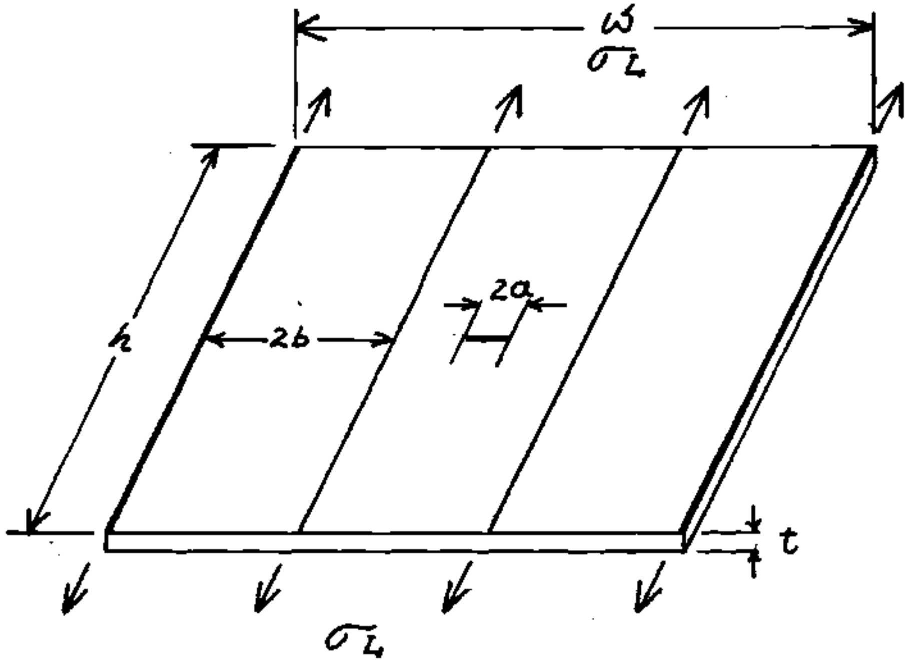

Figure 3-1. Panel Configuration

'F' on the panel consists of a sustained load 'F_1' and a random fatigue load 'F_2'. The probabilistic description of the random load is assumed known. Therefore, the external load 'F' is quantified probabilistically. Also, there is a central crack in the panel whose initiation and growth parameters are estimated beforehand. For the fail-safe design of the panel the inspection frequency (i.e. the number of periodic inspections during the design life) has to be specified. Thus, the design variables are the thickness 't', stringer spacing '2b' and the inspection frequency. The subsequent development of the design problem involves problems pertinent (a) to the modes of failure and (b) to the effect of inspection schedule on one of the modes of failure.

3-3. Modes of Failure

Before proceeding with any design problem a thorough understanding of the feasible modes of failure is a customary necessity. In the present reliability based fatigue design problem two modes of failure have to be considered. One of them is called here 'static failure' in which case the external random load has exceeded the residual strength of the panel at a crack length which is far less than the critical length. The random stress due to the applied external loading 'F' is denoted by 'σ_L'. The residual strength of the panel diminishes as the crack length increases and is denoted by 'σ_R' at a crack length 'a'. Initially when the crack has not started to grow due to fatigue the strength is represented by 'σ_u'. The initial margin of safety is given by the probability that σ_u/σ_L is greater than unity. Subsequently when the fatigue effects set in the crack length increases, reducing the residual strength. The margin of safety then is given by the probability that σ_R/σ_L is greater than unity. This quantity denoted as the 'static reliability, R_s, has to be developed as a function of the random load parameters and residual strength parameters.

The second mode of failure that has to be considered is termed 'dynamic fatigue failure' in this research. Under this condition, the crack length 'a' has exceeded the critical crack length 'a_c' at an external stress σ_L which is less than the residual strength σ_R. The measure of safety for this mode is the probability that the crack length 'a' is less than the critical crack length 'a_c'. This quantity denoted as 'dynamic reliability, R_f' has to be evaluated in terms of the parameters of the random crack initiation and propagation. The next problem is to determine the effect of periodic inspection for fatigue damage with attendant repair on dynamic reliability 'R_f'. The overall structural reliability 'R' has to be expressed in terms of the two components 'R_s' and 'R_f'.

3-4. Design Problem Statement

The problem consists of selecting the optimum material, prescribing the optimum geometry and choosing the optimum maintenance schedule. The variables characterizing the materials are the crack initiation parameters, crack growth parameters and the material constants. Geometry is prescribed by the thickness of the panel 't' and the stringer spacing '2b'. Maintenance schedule involves the selection of the number of inspections 'j' during the design life of the panel. The selection of the optimum values for all these variables requires a suitable objective function in terms of all the variables. The total cost function 'C_T' is a convenient objective function since all the variables manifest a tangible effect. The total cost function is the sum of three costs. The first component is the expected cost of failure. This is the product of the overall probability of failure and the deterministic cost of the panel. The second component is the cost due to the inspection schedule. The third one is the actual cost of the structure. Thus,

$$C_T = P_f \cdot C_f + C_s + j \cdot C_i \tag{3-1}$$

where

P_f is the overall probability of failure

C_f is the cost of failure

C_s is the cost of the structure

j is the number of inspections, and

C_i is the cost of an inspection.

Next, C_T has to be minimized with respect to the material, geometric and maintenance variables. The minimization is subject to the reliability constraint, $R \geq R_{bound}$.

The second problem selected is that of reliability-based fracture critical design of a reusable solid rocket motor case of a space vehicle. In this case the randomness of the external load is removed and only the randomness of initial flaw size is considered.

3-5. Solution Methodology

The following are the ground-lines that are followed in the methodology for the reliability based fail-safe fatigue design of the stiffened panel discussed previously.

1. The solution methodology commences with the development of a stochastic model for fatigue crack initiation and growth. The derivation of the governing differential equations and their solution is presented in Chapter IV.

2. Chapter V proposes an improved numerical integration technique for multiple integration w.r.t. one ind-variable.

3. Chapter VI explains the static reliability 'R_s' and its implications with residual strength, and the external random stresses.

4. The effect of intermittent inspections and repair on dynamic reliability 'R_f' is undertaken in Section 6-5. Then overall probability of failure can be determined as a function of 'R_s' and 'R_f'.

5. The development of the total cost function 'C_T' and the application of the entire procedure to two demonstration problems are undertaken in Chapters VIII and IX.

CHAPTER IV

A STOCHASTIC MODEL FOR FATIGUE DAMAGE

4-1. Fatigue Damage

Fatigue is a progressive failure of a structure which is subjected to repeated, cyclic or fluctuating loads. If a structure is subjected to such a fatigue loading it may fracture at a stress level less than that required to cause failure under static conditions. This phenomenon is known as 'fatigue' which is a primary source of failure in aircraft structured members [20,31]. Many fatigue theories have been postulated to explain the underlying mechanism [112]. Basically what happens is that at locations of a surface imperfection the crack commences because of stress intensity. With subsequent fatigue loading the crack builds up to detectable size under a reliable non-destructive inspection procedure. Any further cyclic loading after this crack initiation causes the crack to grow rapidly to such a magnitude at which the residual strength falls below the external load and failure takes place.

4-2. Mathematical Modeling of Fatigue Process

Deterministic fatigue data analysis:--Classically the fatigue life data is described by S-N curves as shown in Figure 4-1. Here S denotes the stress amplitude and N

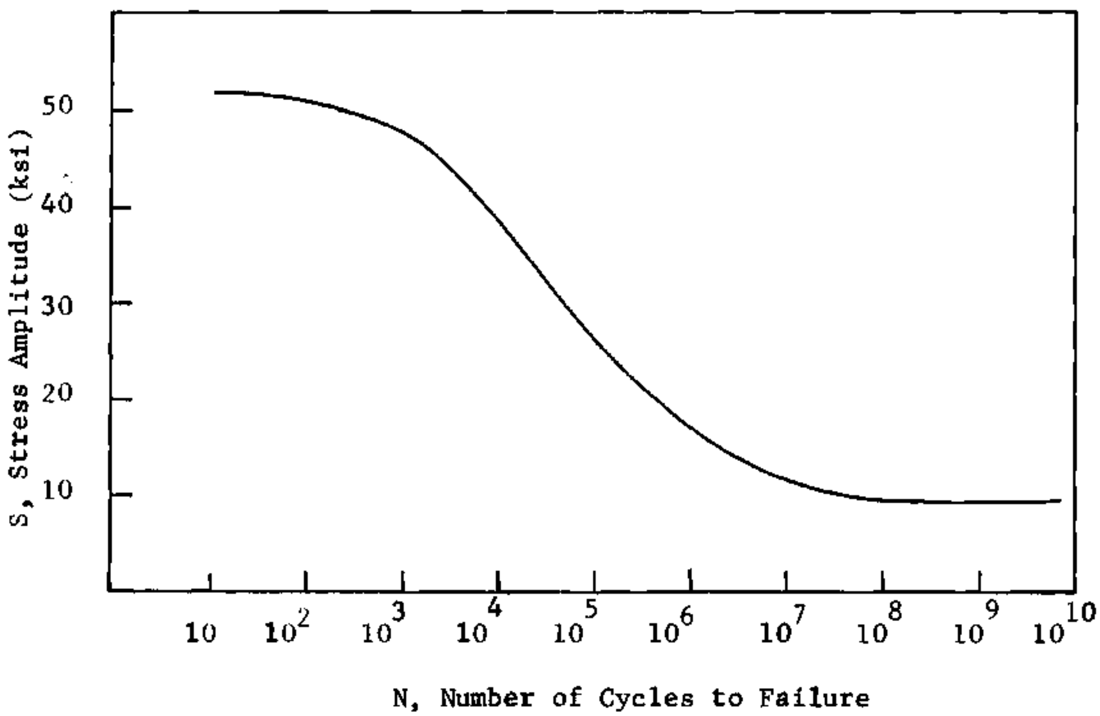

Figure 4-1. Typical S-N Diagram

denotes the number of stress cycles to complete fracture. Plotting these curves with linear scale for S and a logarithmic scale for N is a common practice in engineering. In general, S-N curves represent the progressive structural deterioration due to fatigue loading. The stress level corresponding to the horizontal asymptote on the S-N diagram is designated as the endurance limit. This is a significant quantity for deterministic fatigue design procedures. Several attempts have been made to find general mathematical laws for the relation between stress and number of life cycles. Most of these laws are of empirical nature [103,104].

Another measure of fatigue damage is by means of crack length and its growth rate with the number of load cycles. A number of empirical expressions exists for crack growth rates [112] e.g. Paris, Forman, Colliepriest.

When several identical specimens have been fatigue tested at the same stress level their fatigue lives usually exhibit a great deal of scatter [12]. The S-N curves for these specimens usually fall within a band. The center line of this band is often described as the mean life curve which means that 50 percent of the specimens are expected to fall above and 50 percent below the mean curve. Similarly, curves can be drawn for other survival probabilities, P. These are known as P-S-N curves, Figure 4-2.

When the fatigue loading during the life of the specimen involves several stress amplitudes, the cumulative

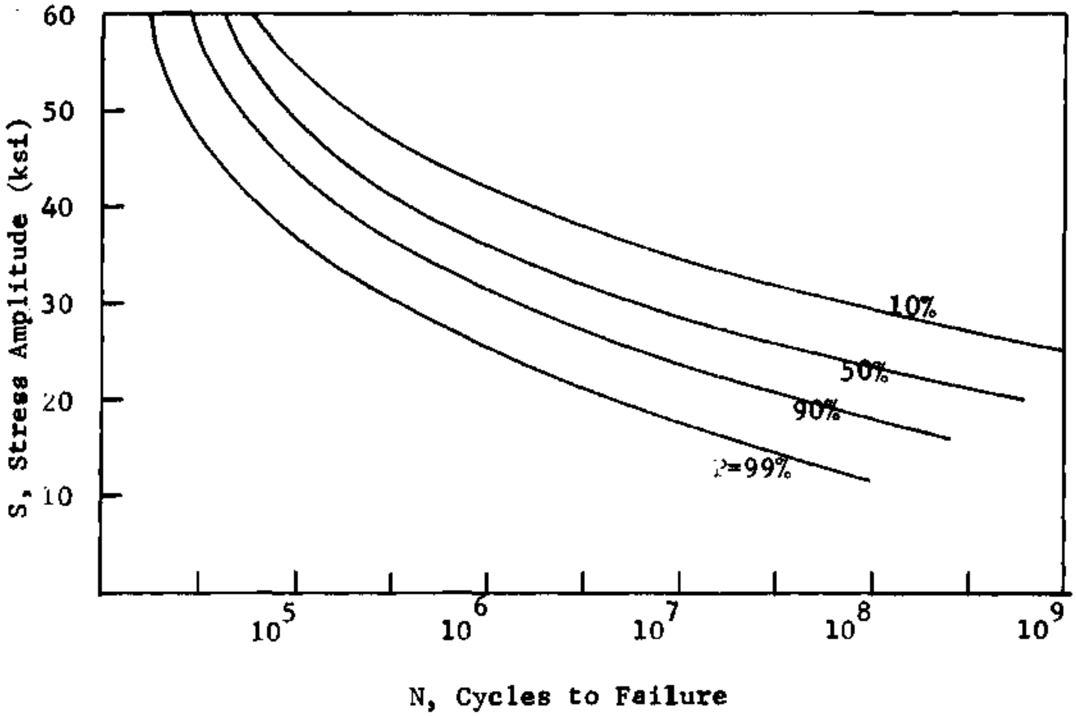

Figure 4-2. A Typical P-S-N Diagram

damage is expressed by rules such as Miner's rule [31] and other similar rules [17]. The effect of mean stress on fatigue life has been expressed by one of several empirical rules such as Goodman's, Gerber's, Gine's [112].

Probabilistic fatigue data analysis:--Many attempts have been made in the past to analyze fatigue data probabilistically. For a group of specimens that are tested to failure at a single stress level the results indicate, in general, that the logarithm of the fatigue life cycles has a Gaussian distribution [105]. Similarly, the fatigue stress level at which a specimen survives a given number of cycles follows a normal distribution [106]. Another approach to modeling is to employ the lowest fatigue life (extreme values) cycles as random variable by using the Weibull frequency distribution [103]. In all these statistical models a single random variable for fatigue life cycles or fatigue strength is considered. The entire fatigue process is described by probabilistic statements such as 'What is the probability that for cycles $n \leq N$ fatigue failure takes place?' Except for the simplicity, the usefulness of such models as Weibull, Log-Normal is limited because they neglect many important aspects of fatigue process.

The inadequacy of these models can be demonstrated by their inability to furnish the following information. For example, one question that needs to be answered is as follows. What is the length of the crack that corresponds to the failure time? Is this the initiation length or critical length or some other arbitrarily chosen length? Initiation length can vary depending upon the available non-destructive inspection (NDI) capability. Furthermore these simple probability models do not provide any information for optimizing the different choices of repair threshold crack lengths, NDI capabilities, crack arresting means etc. Therefore, a single random variable can not describe fatigue process satisfactorily. The reason for this is that fatigue process is a stochastic (random) process in time as described below.

4-3. Fatigue as a Stochastic Process

Fatigue process can be described as an evolutionary stochastic process. A stochastic process is a large ensemble of sample functions which are characterized by probability distributions. What is meant by this is that the crack length 'a' is a stochastic process as a random function of the number of cycles. The justification for this is delineated in Figure 4-3. It is generally accepted that most structural materials are associated with an initial flaw (a_o) distribution. The initial flaw distribution causes the crack initiation time (i.e. the time required for a crack of detectable size by NDI) to be a random variable. Following crack initiation, the subsequent crack growth is a random process because the growth rate, da/dN is random

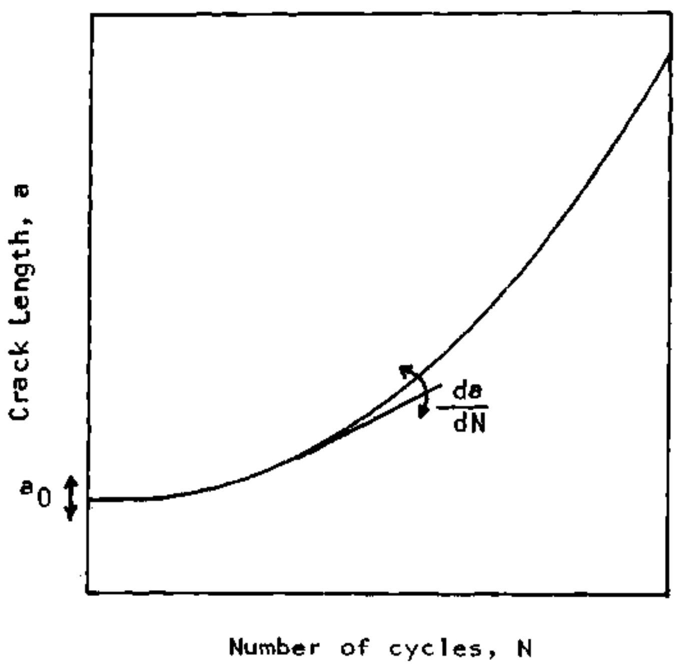

Figure 4-3. Illustration of Randomness in Fatigue Cracks

process. The reason for this is that the external loading is in general, a stochastic process.

Assumptions: For the development of the stochastic model for fatigue crack growth a single crack is assumed to be present at a fatigue critical region as in Figure 3-1. Then, the variation of the crack length, 'a', is qualitatively of the type shown in Figure 4-4. The crack length is considered to be a continuous variable with respect to the time. The corresponding model for the stochastic process for fatigue crack growth involves the consideration of continuous state variable (crack length) and continuous time variable. It is difficult to develop such a model. The development of the model is simplified by considering the state variable as discrete and time as continuous variable as shown in Figure 4-4.

The implication of this assumption is that crack growth takes place in discrete steps, Figure 4-4. This assumption is justified for the following reasons. (1) In actuality, fatigue crack growth under cyclic loading takes place only during the upward (increasing) portion of load cycle. Therefore, the cycle to cycle growth of the crack is best represented by a minute step-wise propagation. (2) For a large number of cycles the crack growth curve appears to be a smooth continuous curve. Then, it is always possible to represent a continuous curve by small discrete steps reasonably accurately. (3) The step size can be made as

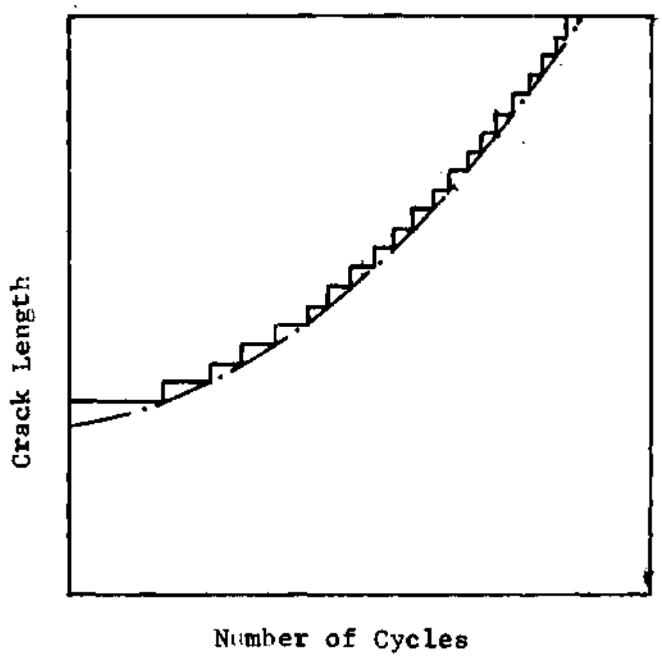

Figure 4-4. Crack Growth with Time

small as possible to approach the continuous growth.

(4) Structural metals are crystalline.

The next assumption is that the random growth of the discrete steps in a small interval of time is assumed to be a Poisson process for the following reasons: (1) The random step units grow with a mean growth rate, (2) In a small interval of time the only two possibilities are either a discrete step grows or it doesn't grow. Then, the simplest process fulfilling these requirements is the Poisson process with only one parameter, namely the mean growth rate. A more complex stochastic process that may be feasible would be Markov process.

The final assumption is that the probability distribution of the time prior to crack initiation is known.

Derivation: Since the crack is assumed to grow in discrete steps, the length 'a' of the crack is expressed as $k \cdot \Delta\ell$, where $\Delta\ell$ is the magnitude of the discrete step and $k = 0, 1, 2, \ldots$. The average growth rate at a crack length of $k \cdot \Delta\ell$ is denoted by ν_k which is a function of $k \cdot \Delta\ell$. The growth rates ν_k introduce the effects of the external random loading into the stochastic model for crack growth. These can be estimated from the theory of fracture mechanics as explained in Chapter VI. The probability that at time 't' the crack length is $k \cdot \Delta\ell$ is denoted as $P(t,k)$. The Poisson process assumption for the growth of '$\Delta\ell$'s in a small interval of time 'Δt' yields the following

$$P(\Delta t,n) = \frac{(\nu_k \Delta t)^n e^{-\nu_k \Delta t}}{n!}, \quad n = 0,1,2,... \tag{4-1}$$

By expanding the exponential function in equation (4-1) and neglecting the higher order terms results in

$$P(\Delta t,n) = \frac{1}{n!} (\nu_k \cdot \Delta t)^n (1-\nu_k \Delta t), \quad n = 0,1,2,... \tag{4-2}$$

At this time the following question is posed. What is the probability that at time $(t+\Delta t)$ the crack length is $k \cdot \Delta \ell$? The length of the crack can be $k \cdot \Delta \ell$ at time $(t+\Delta t)$ in the following $k+1$ mutually exclusive ways.

(1) The crack length is k units of $\Delta \ell$ at time 't' and zero units grow during Δt. The joint probability of these two independent events is given by the product of their individual probabilities, i.e. $p(t,k) \cdot P(\Delta t,0)$.

(2) The crack length is $(k-1)$ units at time t and 1 unit arrives during Δt. The probability of this compound event is $p(t,k-1) \times P(\Delta t,1)$.

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(k+1) The crack length is zero at time t and k units arrive in the time interval Δt. This probability is $p(t,0) \times p(\Delta t,k)$.

Then the probability $p(t+\Delta t,k)$ is the sum of the probabilities of the 'k+1' mutually exclusive events described above, i.e.

$$P(t+\Delta t,k) = \sum_{m=o}^{k} P(t,k-m)P(\Delta t,m) \qquad (4\text{-}3)$$

Next, considering only the first two terms in the equation (4-3) separately from the rest, it can be rewritten as follows:

$$P(t+\Delta t,k) = P(t,k)P(\Delta t,0) + P(t,k-1)P(\Delta t,1) + S_k \qquad (4\text{-}4)$$

In equation (4-4) S_k is given by the following

$$S_k = \sum_{m=2}^{k} p(t,k-m)P(\Delta t,m) \qquad (4\text{-}5)$$

From the fundamental axioms of probability the following inequalities are true.

$$\sum_{m=2}^{k} p(t,k-m) \leq 1 \qquad (4\text{-}6)$$

$$\sum_{m=2}^{k} p(\Delta t,m) \leq 1$$

Then it is reasonable to write the inequality

$$S_k \leq \sum_{m=2}^{k} P(\Delta t,m) \qquad (4\text{-}7)$$

From the assumption of Poisson distribution for the discrete crack arrivals in Δt the following can be shown to be true for small values of Δt

$$\sum_{m=2}^{k} p(\Delta t,m) \to \frac{(\nu_2 \Delta t)^2}{2} \qquad (4\text{-}8)$$

From equations (4-7) and (4-8) it follows that

$$S_k \le \frac{(\nu_2 \Delta t)^2}{2} \qquad (4\text{-}9)$$

or alternatively,

$$S_k = \frac{\zeta(\nu_2 \Delta t)^2}{2}, \quad \zeta \le 1 \qquad (4\text{-}10)$$

From equation (4-2) the following are obtained

$$P(\Delta t,0) = 1-\nu_k \Delta t+0_2(\Delta t)$$

$$= 1-\nu_k \Delta t \qquad (4\text{-}11)$$

$$P(\Delta t,1) = \nu_{k-1}\Delta t \; (1-\nu_{k-1}\Delta t+0_2 \Delta t)$$

$$= \nu_{k-1}\Delta t \qquad (4\text{-}12)$$

Substituting equations (4-10), (4-11) and (4-12) in equation (4-4), it reduces to the following

$$P(t+\Delta t,k) = P(t,k)(1-\nu_k \Delta t) + P(t,k-1)\nu_{k-1}\Delta t$$

$$+ \zeta \frac{(\nu_2 \Delta t)^2}{2} \qquad (4\text{-}13)$$

Rearranging the terms and letting $\Delta t \to 0$,

$$\lim_{\Delta t \to o} \left(\frac{P(t+\Delta t,k)-P(t,k)}{\Delta t}\right) = -\nu_k P(t,k) + \nu_{k-1} P(t,k-1) +$$

$$+ \lim_{\Delta t \to o} \left(\frac{\zeta \, \nu_2{}^2 \Delta t}{2}\right) \qquad (4\text{-}14)$$

Simplifying equation (4-14) yields

$$\frac{\partial P(t,k)}{\partial t} + \nu_k P(t,k) = \nu_{k-1} P(t,k-1),$$

$$k = 2,3,4,\ldots \qquad (4\text{-}15)$$

Equation (4-15) is valid for $k = 2,3,4,\ldots$. The governing differential equation for $k = 1$ (crack initiation) can be derived from a similar procedure as follows.

Crack initiation: From an engineering point of view crack initiation can be understood as the growth of a crack of minimum size which can be readily detected by NDI procedures. In general, this crack size can be taken as $\delta \cdot \Delta \ell$ where δ is some positive constant. Also, the probability distribution of the time prior to this crack initiation is assumed to be given. From this known information the governing differential equation for a fatigue crack of length $\delta \cdot \Delta\ell$ can be derived and solved. For the sake of simplicity, crack initiation here, is assumed to be the development of a crack of length $1 \cdot \Delta\ell$ (i.e. $\delta = 1.0$). For other values of δ the procedure remains the same except that the corresponding initiation probability distribution has to be used. The following is how the governing differential equation for a fatigue crack length $1 \cdot \Delta\ell$ can be derived. All the assumptions made at the beginning of this section are valid here also. Like before it is convenient to consider the probability of having a crack length of $1 \cdot \Delta\ell$ by the time $t+\Delta t$. The crack length can be $1 \cdot \Delta\ell$ at time $t+\Delta t$ in the following two mutually exclusive manners.

(1) A crack of length $1 \cdot \Delta\ell$ is present at time t and no growth occurs during Δt. The probability of these two mutually independent events is given by $P(t,1) \cdot (1-\nu_1 \cdot \Delta t)$.

(2) No crack is present at time t and a unit $\Delta\ell$ grows during Δt. The probability of this is given by $f_T(t) \cdot \Delta t$ where $f_T(t)$ is the probability distribution function of time prior to crack initiation.

Then the probability $p(t+\Delta t,1)$ is given by the sum of the above two mutually exclusive events as follows.

$$P(t+\Delta t,1) = f_T(t)\Delta t + P(t,1)(1-\nu_1 \Delta t) \qquad (4\text{-}16)$$

Rearranging the terms in equation (4-16) and taking the limit as $\Delta t \to o$ it can be shown that,

$$\frac{\partial}{\partial t} P(t,1) + \nu_1 P(t,1) = f_T(t) \qquad (4\text{-}17)$$

Thus equations (4-17) and (4-15) are the governing equations for crack initiation and propagation respectively.

Initial conditions: The solution of each of the equations (4-15) and (4-17) requires one initial condition each. These initial conditions are given by the assumption that initially at $t = \phi$ the probability of having a crack of length $k \cdot \Delta\ell$, $k = 1,2,3,\ldots$ is zero, i.e.

$$P(t,k) = 0, \; t = 0 \qquad k = 1,2,3,\ldots. \qquad (4\text{-}18)$$

4-4. Solution of the Governing Differential Equations

The governing differential equations (4-15) and (4-17) are sequentially coupled first order differential equations with constant coefficients. If the solution for $k = 1$ is known, then the solution for $k = 2$ can be obtained and so on. The differential equation governing the crack initiation can be solved as follows. Rewriting the equation (4-17) with the corresponding initial condition from (4-18),

$$\frac{\partial}{\partial t} P(t,1) + \nu_1 P(t,1) = f_T(t), \qquad (4\text{-}17)$$

$$P(t,1) = 0, \quad t = 0 \tag{4-18a}$$

The 'integrating factor method' of solution is employed conveniently here [108,109]. The integrating factor ϕ is given by

$$\phi = e^{\int \nu_1 dt} = e^{\nu_1 t} . \tag{4-19}$$

The solution of (4-17) is given by

$$\frac{d}{dt} [e^{\nu_1 t} P(t,1)] = e^{\nu_1 t} f_T(t) \tag{4-20}$$

By integrating equation (4-20) w.r.t. 't', the following equation is obtained

$$e^{\nu_1 t} P(t,1) = K + \int e^{\nu_1 t} f_T(t) dt \tag{4-21}$$

where K is the constant of integration. From equation (4-21) it follows that,

$$P(t,1) = e^{-\nu_1 t} [K+J(t)] \tag{4-22}$$

In this equation

$$J(t) = \int_{\tau=o}^{t} e^{\nu_1 \tau} f_T(\tau) d\tau \tag{4-22a}$$

By substituting the initial condition (4-18a) into equation (4-22) yields the following result.

$$0 = K + J(0) \tag{4-23a}$$

or

$$K = -J(0) \tag{4-23}$$

From equation (4-22a), $J(0) = 0$ and hence $K = 0$. Now the solution can be concisely written as follows:

$$P(t,1) = e^{-\nu_1 t} \int_o^t e^{\nu_1 \tau} f_T(\tau) d\tau \tag{4-24}$$

The solution $P(t,1)$ can be evaluated if the functional form for the probability distribution function for initiation $f_T(t)$ is given.

Having known $P(t,1)$, the solution of equation corresponding to $k = 2$ can be obtained. The governing differential equation together with the initial condition are as follows.

$$\frac{\partial}{\partial t} [P(t,2)] + \nu_2 P(t,2) = \nu_1 P(t,1) \tag{4-15a}$$

$$P(t,2) = 0, \quad t = 0 \tag{4-18b}$$

The integrating factor is $e^{\nu_2 t}$ and the solution is given by the following equation.

$$P(t,2) = e^{-\nu_2 t} \int_{o}^{t} \nu_1 e^{\nu_2 \tau_1} P(\tau_1,1) d\tau_1 \qquad (4\text{-}25)$$

By substituting for $P(\tau_1,1)$ from (4-24),

$$P(t,2) \; e^{-\nu_2 t} \int_{\tau_1=o}^{t} \nu_1 e^{(\nu_2-\nu_1)\tau_1} \int_{\tau=o}^{\tau_1} e^{\nu_1 \tau} f_T(\tau) d\tau d\tau_1 \qquad (4\text{-}26)$$

Similarly, it follows that

$$P(t,3) = e^{-\nu_3 t} \int_{\tau_2=o}^{t} \nu_2 e^{(\nu_3-\nu_2)\tau_2} \int_{\tau_1=o}^{\tau_2} \nu_1 e^{(\nu_2-\nu_1)\tau_1} \int_{\tau=o}^{\tau_1} e^{\nu_1 \tau} f_T(\tau) d\tau \; d\tau_1 d\tau_2 \qquad (4\text{-}27)$$

The general solution is given as follows.

$$P(t,k) = e^{-\nu_k t} \int_{\tau_k=o}^{t} \nu_{k-1} \, e^{(\nu_k-\nu_{k-1})\tau_k} \int_{\tau_{k-1}=o}^{\tau_k} \nu_{k-2} \, e^{(\nu_{k-1}-\nu_{k-2})\tau_{k-1}} \cdots$$

$$\nu_1 \, e^{(\nu_2-\nu_1)\tau_2} \int_{\tau_1=o}^{\tau_2} e^{\nu_1 \tau_1} f_T(\tau_1) d\tau_1 d\tau_2 d\tau_3 \ldots d\tau_k, \quad k = 1,2,3,\ldots \qquad (4\text{-}28)$$

Normalization: Equation (4-28) derived in the previous section is valid for k = 1,2,3,...∞. However, in reality the observed crack length does not extend to infinity. Failure of the structure occurs long before then. This implies that the probability P(t,k) needs to be normalized by a sum to a realistic maximum number N_{max}.

The new probability function then, becomes the following.

$$\bar{P}(t,k) = \frac{P(t,k)}{\sum_{k=o}^{N_{max}} P(t,k)}, \quad k=1,2,3,\ldots,N_{max} \tag{4-29}$$

In equation (4-29) it is easy to check the following

$$\sum_{k=o}^{N_{max}} \bar{P}(t,k) = 1.0 \tag{4-30}$$

It should be noted that the bigger the normalization number N_{max}, the more accurate would be the distribution.

4-5. Merits of the Discrete Growth Model

Unlike the simple probability distribution models like Weibull distribution, the new model treats the fatigue phenomenon as a stochastic process. That means, the crack length is a discrete random variable whose probability distribution is continuously changing with time. This is a more realistic description of fatigue process than the simple models where only one random variable is employed to characterize fatigue process.

The model is of great usefulness in making probabilistic statements on crack length with respect to time, whereas crack length is totally absent in the simple fatigue models. For example the maintenance department is interested in knowing the probability of a repair at a critical location before certain number of flight hours, or between two given flights. This can be done by setting a repair threshold crack length and finding the probability of exceeding the repair threshold length for the time period under question. Similarly, the probability of fatigue failure can be expressed as the probability of having a crack length greater than the critical length. Similar statements can be made for crack initiation also. Thus the developed stochastic model encompasses all the three stages of the fatigue phenomenon, namely initiation, growth and failure.

The model can be used to optimize the repair threshold crack length by writing the probability statement for the repair of fatigue cracks. This repair probability can be incorporated in a suitable objective function such as expected total cost function. The total cost function can then be optimized and the optimum threshold crack length for repair of fatigue damage may be selected.

The model greatly facilitates the optimization of the inspection and maintenance schedule. This can be done by writing the probability statement for fatigue failure under a given number of inspections and repairs. Then, once again the total expected cost function can be minimized and the optimum number of inspections may be obtained.

In a similar manner an optimum geometry of the panel structure can be arrived at for a better crack arresting characteristic and thus longer fatigue life.

CHAPTER V

AN IMPROVED NUMERICAL SCHEME TO EVALUATE MULTIPLE INTEGRALS

5-1. Necessity

Multiple integration with respect to one independent variable is encountered on many an occasion in engineering problems. For example, the following multiple integral was developed in Chapter IV:

$$P(t,k) = e^{-\nu_k t} \int_{t_{k-1}=o}^{t} \nu_{k-1} e^{(\nu_k - \nu_{k-1})\tau_k} \quad \cdots$$

$$\int_{\tau_1=o}^{\tau_2} e^{\nu_1 \tau_1} f_T(\tau_1) d\tau_1 d\tau_2 \ \cdots \ d\tau_k$$

$$k = 1,2,\ldots N \tag{5-1}$$

A second example is the problem of lateral vibration of beams. In this case, a fourth order governing differential equation needs to be successively integrated four times. Closed form solutions can be obtained only in special cases. In general, analytical solutions cannot be obtained. In cases where analytical methods are not successful, numerical methods have to be used.

In reference [98], an integrating matrix method for numerical integration has been developed. The developed method has been applied to solve the free vibration problem of beams. In the method an integrating matrix of a chosen degree has been used successively four times to integrate the governing differential equation.

The objective of this chapter is to show that by increasing the degree of the integrating matrix after each integration, the accuracy of the results can be improved. This principle is later used to develop a modified integrating matrix method for numerical integration.

5-2. Mathematical Motivation

In this section the integrating matrix method of reference [98] is reviewed briefly. A multiple integral of the form involving k successive integrations is considered.

$$I = \int_{x_o}^{x_N} \cdots \int_{x_o}^{\bar{x}_3} \cdots \int_{x_o}^{\bar{x}_2} f(\bar{x}_1)\,d\bar{x}_1\,d\bar{x}_2 \ldots d\bar{x}_k \tag{5-2}$$

In the equation, x_o and x_N are the lower and upper limits of integration respectively, and $\bar{x}_1$, $\bar{x}_2$, ..., $\bar{x}_k$ are dummy variables of integration. The range of integration $(x_o - x_N)$ is divided into N equal intervals of width 'h'. Then,

$$x_N = x_o + Nh \tag{5-2a}$$

The values of the integrand $f(x)$ at these $(N+1)$ stations, $f_o, f_1, \ldots, f_N$ are obtained (Figure 5-1).

The function $f(x)$ is approximated by an nth degree polynomial in each of the N intervals, i.e.,

$$f(x) \approx \sum_{i=o}^{n} e_i x^i, \quad \begin{cases} x_{j-1} \leq x \leq x_j \\ 0 < j \leq N \end{cases} \tag{5-3}$$

The first newton interpolation formula (reference 108,109) is used to obtain $f(x)$ for equidistant arguments.

$$f(x) \approx \tilde{f}(t) = f_o + \frac{t}{1!} \Delta_o^1 + \frac{t(t-1)}{2!} + \Delta_o^2 + \ldots +$$

$$\frac{t(t-1)\ldots t-n+1)}{n!} \Delta_o^n + R_n \tag{5-4}$$

In this equation,

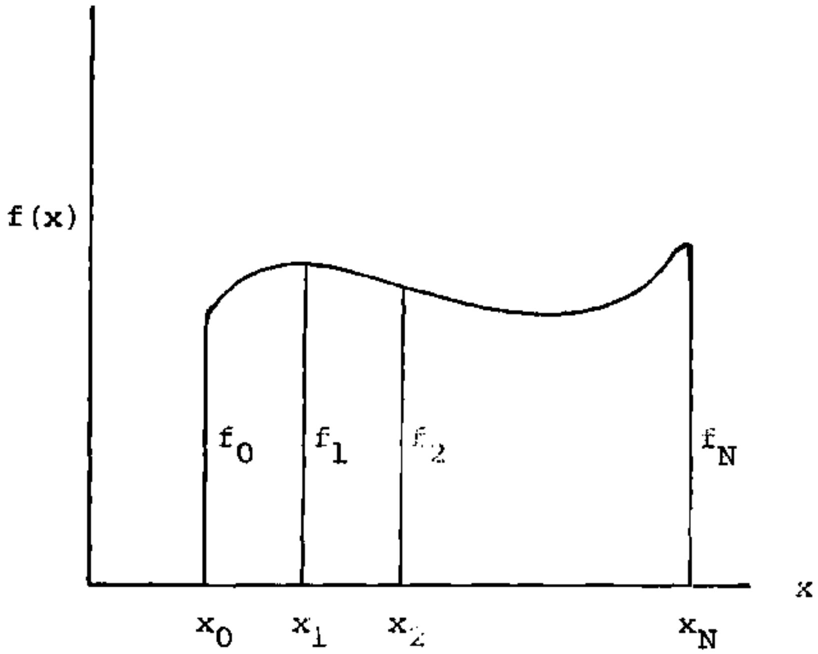

Figure 5-1. Integrand Function

$$\Delta_o^j = f_j - \binom{j}{1} f_{j-1} + \binom{j}{2} f_{j-2} - \ldots (-1)^j f_o$$

$$x = x_o + th \qquad (5\text{-}5)$$

$$R_n = \binom{t}{n+1} h^{n+1} f^{(n+1)}_{(\xi)}, \quad \xi \varepsilon (x_o, x_o+nh)$$

It is to be noted that R_n is identically equal to zero for an nth degree polynomial, because $f(x)^{(n+1)}$ is identically equal to zero. The first integration in the ith interval can be written as follows.

$$\int_{x_{i-1}}^{x_i} f(\bar{x}_1) d\bar{x}_1 \simeq h \int_{i-1}^{i} \tilde{f}(t) dt, \quad i = 1,2,\ldots N \qquad (5\text{-}6)$$

From equations (5-6) and (5-5) it can be shown that

$$\tilde{f}(t) = f_o + t\Delta_o' + \frac{t(t-1)}{2} \Delta_o^2 \qquad (5\text{-}7)$$

for N = n = 2. In this equation

$$\Delta_o^1 = f_1 - f_o,$$

$$\Delta_o^2 = f_2 - 2f_1 + f_o, \qquad (5\text{-}8)$$

$$h \int_{i-1}^{i} \tilde{f}(t)dt = \frac{h}{24} [(24t-18t^2+4t^3)f_o + (24t^2-4t^3)f_1 +$$

$$(4t^3-6t^2)f_2]_{i-1}^{i}, \qquad i = 1,2 \tag{5-9}$$

Then, for i = 1,2 the following equation can be written

$$\int_{x_o}^{x_1} f(\bar{x}_1)d\bar{x}_1 = h \int_0^1 \tilde{f}(t)dt = \frac{h}{12} [5f_o+8f_1-f_2], \tag{5-10}$$

and

$$\int_{x_1}^{x_2} f(\bar{x}_1)d\bar{x}_1 = h \int_1^2 \tilde{f}(t)dt = \frac{h}{12} [-f_o+8f_1+5f_2] \tag{5-11}$$

If N>n, the first equation is repeated (N-1) times, i.e.

$$\begin{aligned}
\int_{x_o}^{x_1} f(\bar{x}_1)d\bar{x}_1 &= \frac{h}{12} (5f_o+8f_1-f_2), \\
\int_{x_1}^{x_2} f(\bar{x}_1)d\bar{x}_1 &= \frac{h}{12} (5f_1+8f_2-f_3), \\
\int_{x_2}^{x_3} f(\bar{x}_1)d\bar{x}_1 &= \frac{h}{12} (5f_2+8f_3-f_4), \\
&\vdots \\
\int_{x_{N-1}}^{x_N} f(\bar{x}_1)d\bar{x}_1 &= \frac{h}{12} (-f_{N-2}+8f_{N-1}+5f_N)
\end{aligned} \tag{5-12}$$

In matrix form this integration can be written as follows.

$$\underset{(N+1)x1}{\{\int} f(x_1)dx_1\} = \underset{(N+1)x(N+1)}{[B]} \; \underset{(N+1)x(N+1)}{[A_2]} \; \underset{(N+1)x1}{\{f\}} \tag{5-13}$$

$$\{\underset{(N+1)x1}{\int} f(x_1)dx_1\} = \left\{\begin{matrix} \int_{x_o}^{x_o} f(\bar{x}_1)d\bar{x}_1 \\ \int_{x_o}^{x_1} f(\bar{x}_1)d\bar{x}_1 \\ \vdots \\ \int_{x_o}^{x_N} f(\bar{x}_1)d\bar{x}_1 \end{matrix}\right\} \tag{5-14}$$

$$\underset{(N+1)x(N+1)}{[B]} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & & \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \tag{5-15}$$

$$\underset{(N+1)x(N+1)}{[A_2]} = \frac{h}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & & 0 \\ 5 & 8 & -1 & 0 & \cdots & & 0 \\ 0 & 5 & 8 & -1 & \cdots & & 0 \\ & \vdots & & & & & \\ 0 & 0 & 0 & 0 & -1 & 8 & 5 \end{bmatrix} \tag{5-16}$$

and

$$\underset{(N+1)x1}{\{f\}} = \begin{Bmatrix} f_o \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_N \end{Bmatrix} \tag{5-17}$$

Finally

$$\{\int f(x)dx\} = [I_2]\{f\} \tag{5-18}$$

where $[I_2] = [B]\ [A_2]$ is called the second degree integrating matrix in reference [98].

5-3. Modification of the Integrating Matrix Method

When equation (5-3) is integrated once an (n+1)th degree polynomial is obtained. In reference [98], this (n+1)th degree polynomial is approximated by an nth degree polynomial. It is obvious from equation (5-5) that R_n will not be zero. Hence, to reduce error for the second integration, the integrand must be approximated by an (n+1)th degree polynomial. This is the improvement suggested in this thesis.

For the second integration, the integrand can be written as follows.

$$G(x) = \int_{x_o}^{x} f(\bar{x}_1)d\bar{x}_1 \tag{5-19}$$

From equation (5-13), on integration the following equation is obtained

$$G(x) = \sum_{i=o}^{n} d_i x^{iH} \tag{5-20}$$

which is an (n+1)th degree polynomial. Analogous to equation (5-4) an expression for G(x) can be written as follows:

$$G(x) = g(t) = g_o + \frac{t}{1!} \Delta_o + \frac{t(t-1)}{2!} \Delta_o^2 + \ldots +$$

$$\frac{t(t-1)\ldots(t-n+2)}{(n+1)!} \Delta_o^{n+1} + R_{n+1} \tag{5-21}$$

By using equation (5-4) and (5-5), it is possible to show that for n = 2,

$$\{\int_{x_o}^{x_i} G(x)dx\} = [I_3]\{G\} \tag{5-22}$$

where $[I_3]$ is the third degree integrating matrix. By substituting equation (5-8) in (5-9),

$$\{\int_{x_o}^{x_i} G(x)dx\} = [I_3][I_2]\{f\} \tag{5-23}$$

Similarly, for k integrations, the following expression is written

$$\int_{x_o}^{x_N} \cdots \int_{x_o}^{\bar{x}_2} f(x)d\bar{x}_1 \ldots d\bar{x}_k = [I_{k+2}][I_{k+1}]\ldots[I_3][I_2]\{f\} \quad (5\text{-}24)$$

5-4. Examples

The comparison of the results of the classical integrating matrix method and those from the improved integrating matrix method are illustrated by using three examples where the exact solution is known.

1. Multiple Integration of an Algebraic Function

As a first example the following integration is considered

$$y = \int_o^{20} \int_o^{x} \int_o^{x} \int_o^{x} f(\bar{x})d\bar{x}\; d\bar{x}\; d\bar{x}\; d\bar{x} \qquad 0 \_ x \_ 20 \qquad (5\text{-}25)$$

In this equation, $f(x) = 1.0$, which is the simplest polynomial that can be considered. By starting with a second degree polynomial for the function, this integration can be formally written as follows

$$\{y\} = [I_5][I_4][I_3][I_2]\{1.0\} \qquad (5.26)$$

This expression corresponds to the modified integrating matrix procedure. If the degree of the integrating matrix is not increased by one after each integration, i.e., the classical integrating matrix procedure of reference [98] is used, the expression for y is as follows

$$\{\bar{y}\} = [I_2][I_2][I_2][I_2]\{1.0\} \tag{5-27}$$

The exact solution of equation (5-25) is easily obtained and can be written:

$$y_{exact} = \left[\frac{x^4}{4!}\right]_0^{20} \tag{5-28}$$

A graphical representation of the error in y, i.e., {y} or $\{\bar{y}\}$ compared to y_{exact} is shown in Figure 5-2. The error is defined as in (12).

$$\%error = 100 \times (y - y_{exact})/y_{exact} \tag{5-29}$$

It can be seen that the percent error ranges from 200.0 at 1/20th division to 0.5 at 20/20th division. The improved method is employed with the same number of divisions (N=20) but with matrices of degree 2, 3, 4 and 5 successively. The percent error is zero all through the range of integration (Figure 5-2). The actual values of the integrals for both cases are delineated in Figure 5-3.

2. Nonhomogenous Differential Equation

As a second example, forced vibration response of a

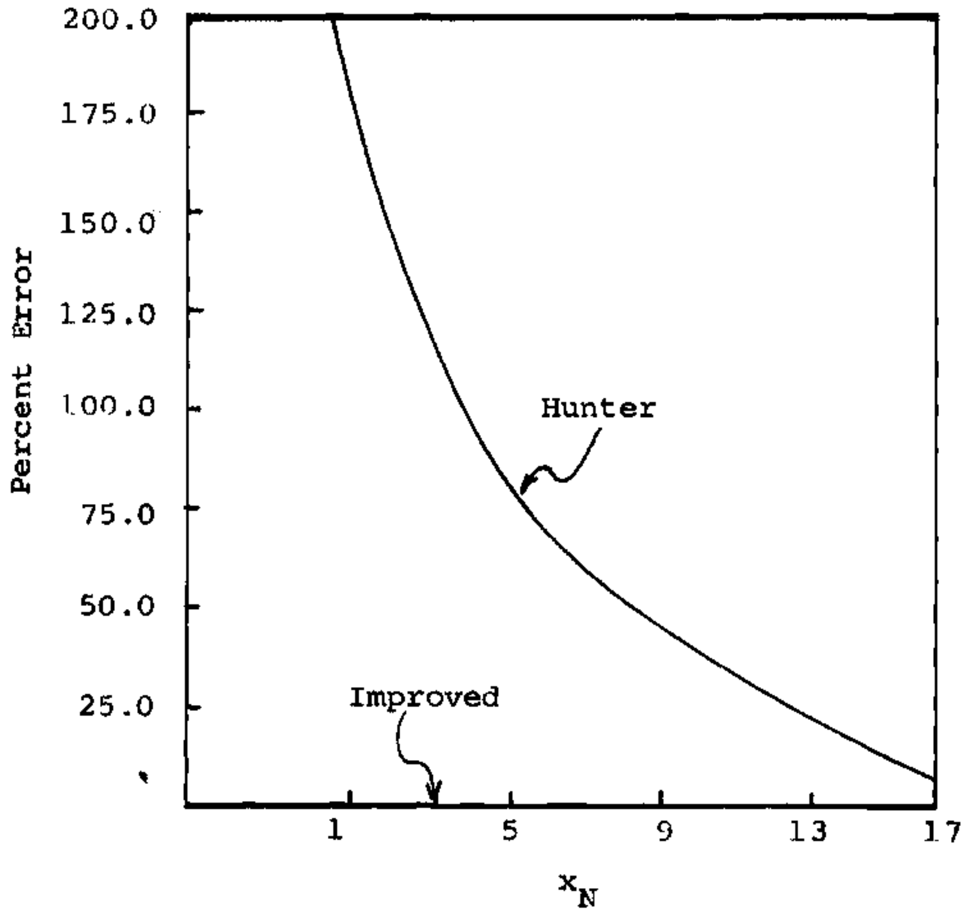

Figure 5-2. Integration of Algebraic Function

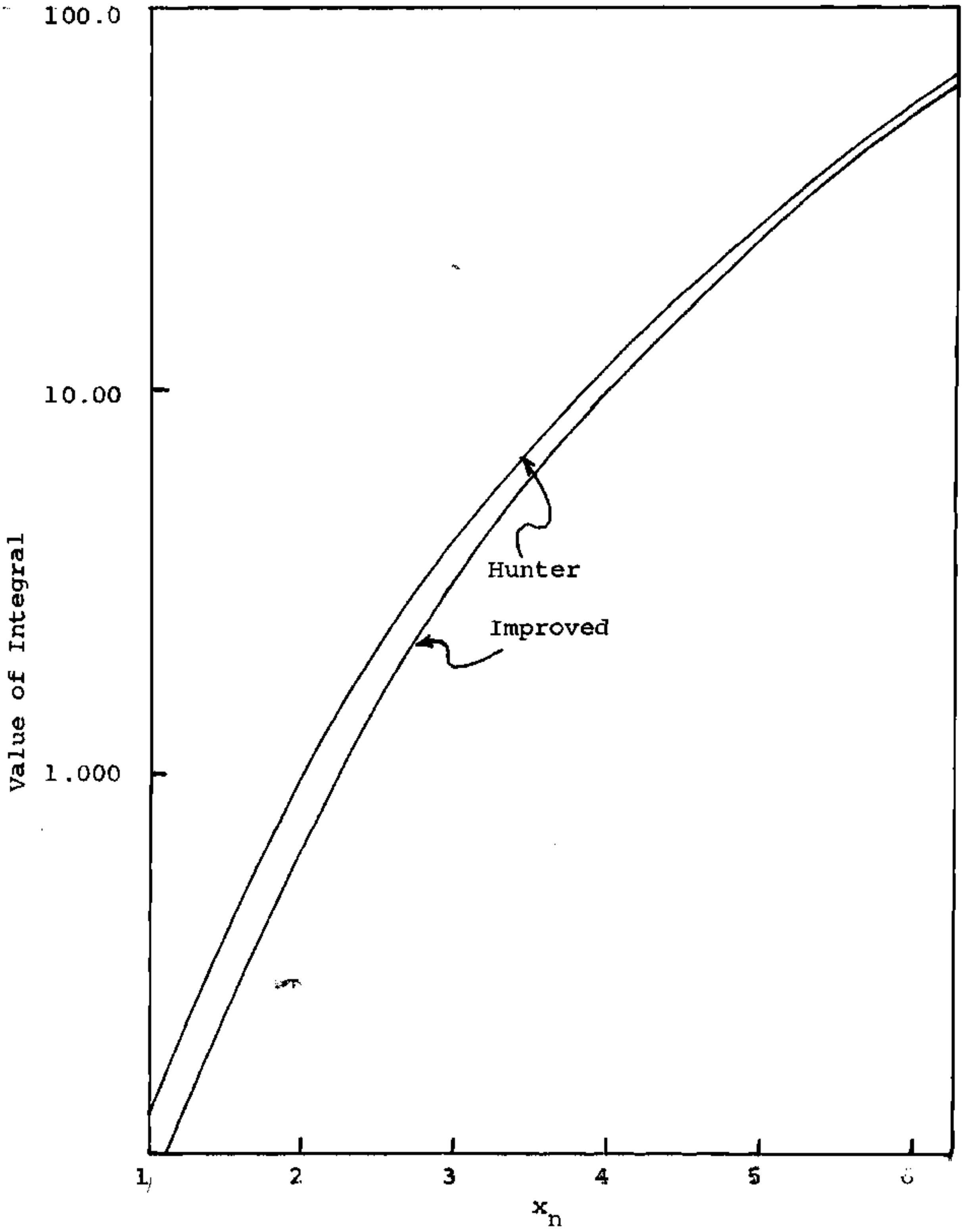

Figure 5-3. Integration of Algebraic Function

cantilever beam as shown in Figure 5-4 is solved by the improved integrating matrix method and the classical method of reference [98].

The governing differential equation [4] for this problem is as follows:

$$EI \frac{{}^{4}y}{x^{4}} + \frac{w}{g} \frac{{}^{2}y}{t^{2}} = F \sin\omega t \qquad (5\text{-}30)$$

In this equation, E is the Young's modulus of the homogeneous, isotropic material of the beam, I is the moment of inertia of the uniform cross section of the beam, y is the lateral deflection of the beam and (w/g) is the mass of the beam per unit length.

The boundary conditions for the problem are given by the known deflection, slope, bending moment and shear force at the respective ends of the cantilever beam. For example, the slope and deflection at the left hand end of the beam are zero. Also, the bending moment and shear force at the right hand end are zero.

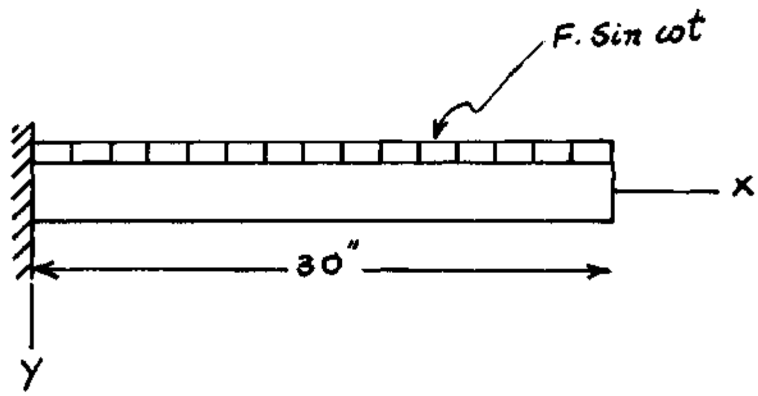

Figure 5-4. Sinusoidally Loaded Centilever Beam

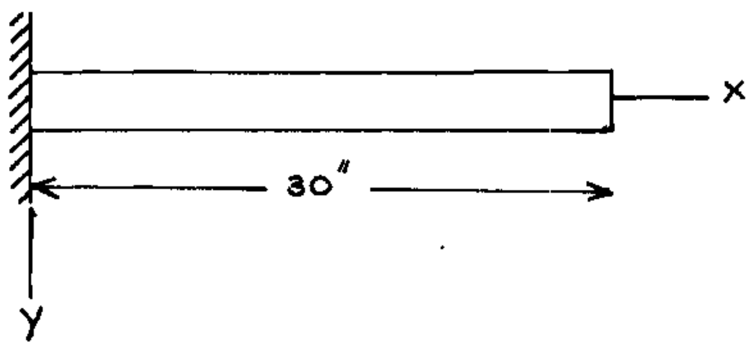

Figure 5-5. Free Vibration of Cantilever Beam

$$y(0,t) = 0$$

$$\frac{\partial}{\partial x}\, y(0,t) = 0$$

$$\frac{\partial^2}{\partial x^2}\, y(\ell,t) = 0 \qquad (5\text{-}31)$$

$$\frac{\partial^3}{\partial x^3}\, y(\ell,t) = 0$$

The exact solution of equation (5-30) satisfying equations (5-31) can be verified to be the following.

$$y(x,t) = [-\frac{F}{EIp^4} + C_1 \sin px + C_2 \cos px + C_3 \sinh px$$

$$+ C_4 \cosh px] \sin \omega t \qquad (5\text{-}32)$$

In equation (5-32) the various constants are given by the following

$$p^4 = w\omega^2/EIg$$

$$C_1 = \frac{F}{DEIp^4} [\cos p\ell \sinh p\ell + \cosh p\ell \sin p\ell]$$

(5-33) continued

$$C_2 = - \frac{-F}{DEIp^4} (\sin p\ell \sinh p\ell - \cosh p\ell \cos p\ell - 1)$$

$$C_3 = - \frac{F}{DEIp^4} (\cos p\ell \sinh p\ell + \sin p\ell \cosh p\ell)$$

$$C_4 = \frac{F}{DEIp^4} (\cos p\ell \cosh p\ell + \sin p\ell \sinh p\ell + 1) \tag{5-33}$$

$$D = 2 (1+\cos p\ell \cosh p\ell)$$

Numerical Solution: A separable type of solution is assumed as follows:

$$y(x,t) = x(x) \sin\omega t \tag{5-34}$$

Equation (5-34) is substituted into equation (5-30) with the following resulting equation.

$$EI \frac{d^4x}{dx^4} - \frac{w\omega^2}{g} x = F \tag{5-35}$$

Equation (5-35) can be written in matrix form as follows.

$$EI \frac{d^4}{dx^4} \{x\} - \frac{w\omega^2}{g} \{x\} = \{F\} \tag{5-36}$$

By multiplying equation (5-36) successively with $[I_2]$, $[I_3]$, $[I_4]$ and $[I_5]$ and using the boundary conditions in the appropriate matrix form it can be shown that

$$\{x\} = \frac{F}{EI}[P_5]^{-1}[P_4]\{1\} \tag{5-37}$$

In this equation,

$$
\begin{aligned}
[P_5] &= [[\ 1\ ] - \frac{w\omega^2}{EIg}[P_4]] \\
[P_4] &= [[I_5] - \{1\}[B_2][I_5]][P_3] \\
[P_3] &= [[I_4] - \{1\}[B_2][I_4]][P_2] \\
[P_2] &= [[I_3] - \{1\}[B_1][I_3]][P_1] \\
[P_1] &= [[I_2] - \{1\}[B_1][I_2]] \\
[B_1] &= [000\ \ldots\ 1] \\
[B_2] &= [100\ \ldots\ 0]
\end{aligned} \tag{5-38}
$$

Then the $\{y\}$ matrix is given by

$$\{y\} = \{x\}\sin\omega t \tag{5-39}$$

If the degree of integrating matrix is not increased by one after each integration then, in equation (5-38)

$$[I_5]=[I_4]=[I_3]=[I_2] \tag{5-40a}$$

The corresponding solution is given by $\{\bar{y}\}$ as follows

$$\{\bar{y}\} = \{\tilde{x}\} \sin\omega t \tag{5-40}$$

The error in equations (5-39) and (5-40) is compared to the exact solution given by equation (5-32) and is graphically represented in Figure 5-6.

Results: For the forced vibration problem the span is divided into 5 equal intervals (N=5) and a second degree integrating matrix is employed four times consecutively. The percent error ranges from 6.4 at 1/5 span to 0.3 at 5/5 span. The improved technique with the same N=5 but with increasing degree of integrating matrix (i.e. I_2 to I_5) gives a maximum percent error of only 0.03 (Figure 5-6).

3. Eigenvalue Problem

The problem is that of evaluating the free vibration characteristics of a cantilever beam.

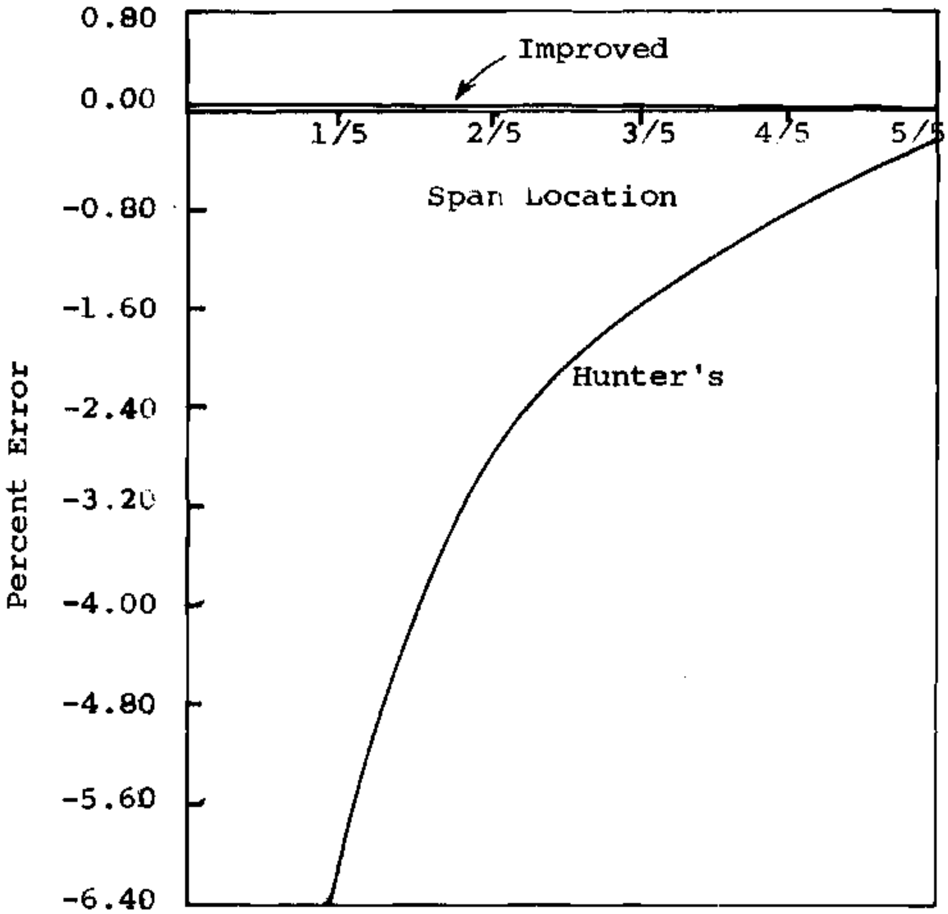

Figure 5-6. Forced Vibration Problem

By letting F = 0 in example (2) the following eigenvalue problem is obtained.

$$\frac{1}{\omega^2}\{x\} = \frac{w}{EIg}[P_5]\{x\} \tag{5-41}$$

where $[P_5]$ is given in equation (5-38). Now using the sweeping matrix technique and matrix iteration, the first, second and the third mode shapes and the corresponding natural frequencies are obtained.

The same procedure is repeated for the case in which the degree of integrating matrix is not increased. The results are listed in Tables 5-1 and compared with the exact solution [105].

5-4. Results

For the free vibration problem percent error in natural frequency and mean square error in mode shape are compared in both techniques. Percent error and mean square error are defined as follows.

$$\begin{array}{l}\text{Mean square error}\\ \text{in mode shape}\end{array} = \sum_{i=1}^{N} \frac{(\text{Exact-Numerical})^2}{N} \tag{5-42}$$

$$\begin{array}{l}\text{\% error in}\\ \text{natural frequency}\end{array} = \frac{\text{Exact-Numerical}}{\text{Exact}} \times 100 \tag{5-43}$$

For the improved technique the degree of the integrating matrix is increased from 2 whereas it is kept constant for

Table 5-1. First, Second and Third Mode Results

N	Improved Method		Method in Ref. [101]	
	Mean Square Error	% Error in Natural Frequency	Mean Square Error	% Error in Natural Frequency
	5-1a. First Mode Results			
4	1.0×10^{-7}	0.42	2.6×10^{-5}	1.84
5	0.00	0.13	1.0×10^{-7}	0.04
7	0.00	0.10	0.00	0.10
7	0.00	0.10	0.00	0.10
	5-1b. Second Mode Results			
4	0.27	3.30	0.63	27.0
5	0.01	1.65	0.14	1.26
7	0.16×10^{-4}	1.63	5.00×10^{-4}	0.91
9	0.09×10^{-4}	1.65	170×10^{-4}	1.38
15	1.70×10^{-4}	1.64	4.0×10^{-4}	1.60
15	1.70×10^{-4}	1.64	4.0×10^{-4}	1.60
	5-1c. Third Mode Results			
15	No shape		No shape	
13	No shape		No shape	
15	0.02	0.04	No shape	
20	0.3×10^{-4}	0.003	No shape	
30	0.2×10^{-4}	0.002	0.20×10^{-2}	1.50
50	0.05×10^{-4}	0.002	0.60×10^{-4}	0.003

the method in Reference [98]. Less than one percent error in first mode is obtained for N = 4 in the improved method whereas N = 5 for the other method to get the same accuracy. The corresponding mean square errors are 1.0×10^{-7} and 2.64×10^{-5}.

For the second mode with N = 13, improved method gave a mean square error of 0.9×10^{-4} where as the other method [98] gave 15.0×10^{-4}. The natural frequency in both the cases is roughly the same.

For the third mode N = 15 was sufficient to get a percent error in natural frequency of 0.023 and 1.6×10^{-4} mean square error whereas N = 30 was necessary for the other method to get a comparable accuracy.

5-5. Conclusions

The numerical examples worked out in this chapter ratify that the accuracy is greatly improved by increasing the degree of the integrating matrix after each integration. This is true for any given number of intervals of division and for any starting degree of the integrating matrix. The difference between the two methods decreases as the number of intervals of the division and the starting degree of integrating matrix increase.

In the case of the free vibration problem, less than 1 percent error in natural frequency and/or less mean square error is obtained at a lower number of spanwise divisions than in the case when the integrating matrix is not altered. Also the mean square error in the mode shape compared to the exact mode shpe for any mode is less in the improved method than the method in Reference [98].

Now it is possible to integrate the expression for $P(t,k)$, the probability of crack being of length $k \cdot \Delta\ell$ at time t.

CHAPTER VI

FRACTURE MECHANICS AND RELIABILITY CONSIDERATIONS

In Chapter III the problem setting for the reliability based, fail-safe design of a stiffened panel with a central crack, under random loading (Figure 3-1) is presented. There, the sequence of solution methodology is also given. In accordance with that solution methodology, a stochastic model for fatigue crack growth is developed in Chapter IV. This model involves the mean crack growth rates 'ν_k' for the single central crack present in the stiffened panel (Figure 3-1) under study. The parameters 'ν_k' are derived from Fracture Mechanics theory considerations in Sections 6-1 and 6-2.

6-1. Residual Strength

The intensity of stress field in the vicinity of the crack tip of an unstiffened sheet is governed by the stress intensity factor K_u given below [18]

$$K_u = \sigma\sqrt{\pi a \quad f(\tfrac{a}{w})} \tag{6-1}$$

In this equation σ is the uniform in-plane stress which acts on the sheet in a direction perpendicular to the direction of the crack growth, 'a' is the crack length, $f(\frac{a}{w})$ is the finite width correction factor and 'w' is the width of the sheet. The effect of the stiffeners is to reduce the stress at the tip of the crack according to the tip reduction factor C_R, defined below [23]

$$C_R = \frac{K_s}{K_u} \qquad (6\text{-}2)$$

'K_s' in equation (6-2) is the stress intensity factor for the stiffened sheet. From equations (6-1) and (6-2) the stress intensity factor for the stiffened sheet is given as follows

$$K_s = C_R \sigma \sqrt{\pi a \quad f(\frac{a}{w})} \qquad (6\text{-}3)$$

Now, assuming that failure occurs when K_s has attained a value equal to the plane stress fracture toughness K_{1c}, the residual strength σ_{Rs}, curve is given by the following.

$$\sigma_{Rs} = K_{1c}/C_R \sqrt{\pi a_c \quad f(\frac{a_c}{w})} \qquad (6\text{-}4)$$

In this equation σ_{Rs} is the residual strength, 'a_c' is the critical half-crack length and 'w' is the width of the sheet. Now, mean crack growth rate can be found as follows.

6-2. Mean Crack Growth Rate

A knowledge of the deterministic crack growth rate is essential to determine the mean crack growth rate parameters

'ν_k'. Reference [22] has presented an empirical growth law which considers both constant and variable amplitude fatigue loading. This can be written as follows.

$$\frac{da}{dN} = \frac{C_1(\Delta K)^n}{(1-r)K_{1c} + \Delta K} \tag{6-5}$$

In equation (6-5), C_1 and n are material constants, ΔK is the range of stress intensities, K_{1c} is the plane stress fracture toughness, 'r' is the ratio of minimum and maximum stress intensity factors, 'a' is the half-crack length and 'N' is the number of cycles. The range of stress intensities ΔK for a stiffened panel can be written as follows.

$$\Delta K = \Delta L \sqrt{f(\tfrac{a}{w})\pi a} \quad C_R(a,b) \tag{6-6}$$

In this equation ΔL is the range of applied loads at a given number of cycles 'N' and 'b' is half of the stringer spacing. For a given value of crack length, say $a = a_1$, $(\frac{da}{dN})$ is a function of the random load parameters 'ΔL' and 'r'. Therefore, for a given crack length ($a = a_1$) the growth rate V_k is a random variable defined by some stochastic process if $\Delta L(N)$ and $r(N)$ are assumed to be defined by stationary stochastic processes. If their second order joint distribution $f(r,\Delta L)$ is given, then, the expected crack growth rate is given as follows.

$$E\left[\frac{da}{dN}\right]_{a=a_1} = \int_{R_{\Delta L}} \int_{R_r} \left[\frac{da}{dN}\right]_{a=a_1} f(r,\Delta L)\,dr\,d(\Delta L) \qquad (6\text{-}7)$$

where '$R_{\Delta L}$' and 'R_r' are the range spaces of ΔL and r, respectively. Equation (6-7) gives the mean crack growth rate at any crack length under the random loading process. This quantity expressed in terms of the discrete length units '$\Delta \ell$' is required in the expression for p(t,k), equations (4-22).

The mean crack growth rate as given by (6-7) in general, does not have a closed form solution because of the complicated integration. An approximate solution can be found by a multi-dimensional Taylor series expansion of (da/dN) in 'ΔL' and 'r' about their mean values $\mu_{\Delta L}$ and μ_r as given by reference [3].

From the theory of multivariate approximations, the expected value of a function $g(X_1,X_2)$ of two independent random variables X_1 and X_2 is given by the following equation [3]

$$E[g(x_1,x_2)] \approx g(\mu_{x_1},\mu_{x_2}) + \frac{1}{2}\frac{\partial^2 g}{\partial x_1^2}\,\sigma_{x_1}^2$$

$$+ \frac{\partial^2 g}{\partial x_1 \partial x_2}\,\rho_{x_1,x_2}\,\sigma_{x_1}\sigma_{x_2} + \frac{1}{2}\frac{\partial^2 g}{\sigma x_2^2}\,\sigma_{x_2}^2 \qquad (6\text{-}8)$$

In this equation,

$E[\cdot]$ = expected value of $(\cdot)$

$\sigma_{\cdot}^{2}$ = variance of $(\cdot)$

$\mu_{\cdot}$ = mean of $(\cdot)$, and

$p_{\cdot,*}$ = correlation coefficient of $(\cdot)$ and $(*)$

By applying equation (6-8) to obtain the mean growth rate i.e. $E[\frac{da}{dN}]$, the result is as follows.

$$E[\frac{da}{dN}] = [A\cdot\mu_{\Delta L}^{n} + \frac{1}{2}\,\sigma_{\Delta L}^{2}\cdot A\cdot n(n-1)\,\mu_{\Delta L}^{2}]D^{-1} +$$

$$[\rho_{\Delta L,r}\cdot\sigma_{\Delta L}\cdot\sigma_{r}\cdot A\cdot n\cdot K_{1c}\cdot\mu_{\Delta L}^{n-1} - \sigma_{\Delta L}^{2}\cdot A\cdot B\cdot n\cdot\mu_{\Delta L}^{n-1}]D^{-2}$$

$$+ [\frac{1}{2}\cdot\sigma_{\Delta L}^{2}\cdot A\cdot B^{2}\cdot\mu_{\Delta L}^{n} - 2\rho_{\Delta L,r}\cdot\sigma_{\Delta L}\cdot\sigma_{r}\cdot A\cdot B\cdot K_{1c}\mu_{\Delta L}^{n}$$

$$+ A\cdot\sigma_{r}^{2}\cdot K_{1c}^{2}\cdot\mu_{\Delta L}^{n}]D^{-3} \qquad (6\text{-}9)$$

In this equation,

$$A = C_1 \cdot \{F(a_1,b)\}^{n}$$

$$B = F(a_1,b) \qquad (6\text{-}10)$$

$$F(a_1,b) = \sqrt{\pi a_1\ f\ (\frac{a_1}{w})}\ C_R(a_1,b)$$

and

$$D = (1-\mu_r)K_{1c} + B_{\cdot\mu\Delta L}$$

Now, having known the mean crack growth rate parameters ν_k equations (4-22) can be evaluated. The next step in the reliability based design procedure is to determine the probability of failure of the stiffened panel. This is undertaken in the next section.

6-3. Probability of Failure

The fracture of a stiffened panel structure under fatigue loading is assumed to be governed by the stress intensity factor K_s as defined by equation (6-3). Failure takes place when the value of K_s exceeds the value of the plane stress fracture toughness K_{1c}, i.e.

$$K_s \geq K_{1c} \qquad (6\text{-}11)$$

The crack length 'a' as well as the corresponding residual strength of the panel have probabilistic descriptions at any given number of cycles, N. For example, at an inspection period, say N1 cycles, failure of the panel can take place if the crack length 'a' is greater than or equal to the selected critical crack length 'a_{cr}' as explained in Chapter III. This critical crack length may be selected for reasons other than the structural strength. Another option is that this may be the longer crack length that can be allowed on the basis of specifications or deterministic length. Options are also available not to consider this mode of failure. This event E_f is denoted as critical crack length fatigue failure in this thesis. The critical crack length probability of this event $P[E_f]$ is readily obtained by means of the developed stochastic model.

Another failure event is possible for the same inspection period, N1 cycles. When crack length 'a' is less than 'a_{cr}' failure can take place if the external random load 'σ_L' is greater than the random residual strength 'σ_{Rs}' of the panel. This failure event E_s is designated as 'mode E_s failure' in this thesis. The corresponding probability of failure $P[E_s]$ can be written as follows.

$$P[E_s] = P[N1, \sigma_L > \sigma_{Rs}] \tag{6-12}$$

This probability can be estimated following the method given in reference [55]. In equation (6-12) σ_L is the external applied load which is assumed to be a stationary stochastic process. The residual strength 'σ_{Rs}' is functionally related to the crack length as given by equation (5-4). In order to evaluate equation (6-12) the probabilistic measures such as mean and variance of both σ_L and σ_{Rs} for N1 cycles are needed [55]. A direct application of this method of reference [55] yields the following result.

$$P[E_c] = \frac{n_c^2[(1+V_S^2)(1+3V_s^2)-k^*(k^*-2)(1+V_s^2)^2]}{[k^* n_c(1+V_s^2)-1]^2} \quad (6\text{-}13)$$

In equation (6-13) the various quantities are given as follows.

$$k^* = \frac{n_c[(1+V_S^2)(1+3V_s^2)/(1+V_S^2)]-1}{[n_c(1+V_s^2)-1]} \quad (6\text{-}14)$$

$$n_c = \bar{\sigma}_{Rs}/\bar{\sigma}_L \quad (6\text{-}15)$$

$$V_s{}^2 = var(\sigma_{Rs})/\bar{\sigma}_{Rs}{}^2 \quad (6\text{-}16)$$

$$V_s{}^2 = var(\sigma_L)/\bar{\sigma}_L{}^2 \quad (6\text{-}17)$$

$$\bar{\sigma}_{Rs}, \bar{\sigma}_L = \text{mean of } \sigma_{Rs}, \sigma_L$$

$$var(\sigma_{Rs}),\ Var(\sigma_L) = \text{variance of } \sigma_{Rs}, \sigma_L$$

The mean and variance of σ_{Rs} are obtained as follows. If $Y = g(x_1, x_2, \ldots x_n)$ then, the second order approximation to the mean of Y is given as follows [3]

$$\bar{Y} = g(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 g}{\partial x_i \partial x_j}\right]_{\bar{x}_i,\bar{x}_j}\mathrm{cov}(x_i,x_j) \quad (6\text{-}18)$$

Also, the second order approximation to the variance of Y is given by the following expression [3]

$$\mathrm{Var}[Y] = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial g}{\partial x_i}\right]_{\bar{x}_i}\left[\frac{\partial g}{\partial x_j}\right]_{\bar{x}_j}\mathrm{cov}(x_i,x_j) \quad (6\text{-}19)$$

Employing equations (6-18) and (6-19) to equation (6-4) the mean and variance of σ_{Rs} are obtained as follows

$$\bar{\sigma}_{Rs} = \frac{K_{1c}}{C_R f\sqrt{\pi \bar{a}_{cr}}} + \frac{3}{8}\,\frac{K_{1c}}{C_R f\sqrt{\pi}(\bar{a}_{cr})^{5/2}}\,\mathrm{var}(a_{cr}) \quad (6\text{-}20)$$

$$\mathrm{Var}(\sigma_{Rs}) = \frac{1}{4}\,\frac{K_{1c}^2}{C_R^2 f^2 \pi (\bar{a}_{cr})^3}\,\mathrm{Var}(a_{cr}) \quad (6\text{-}21)$$

The external stress σ_L is taken as follows

$$\sigma_L = \frac{L_{max}}{t} \quad (6\text{-}22)$$

where in equation (6-22) L_{max} is the maximum peak load per unit breadth and t is the thickness. Now the mean and variance of σ_L can be obtained employing equations (6-18) and (6-19) as given below.

$$\bar{\sigma}_L = \frac{\bar{L}_{max}}{t} \tag{6-23}$$

$$Var(\sigma_L) = \frac{Var(L_{max})}{t^2} \tag{6-24}$$

Mean and Variance of Applied Load, L_{max}

In the previous analysis the loading is characterized by the random variables ΔL and r, which are given as follows:

$$r = \frac{L_{min}}{L_{max}} \quad \text{and} \quad \Delta L = L_{max} - L_{min} \tag{6-25}$$

From equation (6-25) it is easy to see that

$$L_{max} = \Delta L/(1-r) \tag{6-26}$$

The mean $\mu_{L_{max}}$ and variance $\sigma_{L_{max}}^2$ can be obtained by employing equations (6-18) and (6-19) as follows:

$$\mu_{L_{max}} = \frac{\mu_{\Delta L}}{1-\mu_r} + \frac{\mu_{\Delta L}\sigma_r^2}{(1-\mu_r)^3}$$

$$\sigma_{L_{max}}^2 = \frac{\sigma_{\Delta L}^2}{(1-\mu_r)^2} + \frac{\mu_{\Delta L}^2\sigma_r^2}{(1-\mu_r)^4} \tag{6-27}$$

Now all the quantities in equation (6-13) are known and the probability of noise E_s fatigue failure $P[E_s]$ can be calculated. It is obvious that the events $[E_f]$ and $[E_s]$ are mutually exclusive. Then, the total probability of failure P_f is given by the sum of their probabilities, i.e.

$$P_f = P(E_f) + P(E_s) \tag{6-28}$$

The next step is to consider the effect of inspections, on P_f. As a first step, the probability of fatigue failure $P(E_f)$ and the effect of inspections and repairs on $P(E_f)$ is undertaken in the following section.

6-4. Fail-Safe Considerations and Effect of Inspections on Probability of Fatigue Failure

Fail-Safe Design

The major design goal in fail-safe design of aircraft is to ensure that any fatigue damage that is incurred during the service life can be detected before the strength of the aircraft falls below an acceptable level. Fatigue damage in the form of cracks in the panels of the aircraft structure develops long before the structural life expires. Safety, then, demands a structural design such that sufficient strength is retained even under the presence of the cracks. Furthermore, it is required that these cracks are detected before they grow to the length at which residual strength is reduced below that at which catastrophic failure takes place. Structures fulfilling these demands are considered fail-safe [17-19]. Some of the essential considerations in fail-safe design are the non-destructive inspection capability, establishment of inspection intervals and the initial margin of safety. Many fail-safe design procedures have been proposed to achieve these goals [17-19]. Practically all the proposed methods are based on deterministic approaches. However, the cracks or minor fatigue damage which dictate the necessity of a fail-safe design procedure are the result of causes that have uncertainties, such as external loads, material imperfections, etc. These uncertainties are better described probabilistically rather than camouflaged by a deterministic factor of safety.

Fail-safety through inspection and maintenance procedures is undertaken in this section. In Figure (6-1a), curve I qualitatively represents crack growth with time or cycles. In this figure, 'T_D' is the intended design life of the structure, at the end of which the critical crack length is expected. However, owing to the uncertainties in initial crack length and growth rates, the critical crack length may be reached (curve II) much earlier than T_D at T_i. Thus, if periodic inspection with attendant maintenance is performed, fail-safety can be achieved.

Probability of Failure under no Inspection

The probability of failure when no inspections are performed can be obtained in the following way. The design life is assumed to be t_D hours. If inspection and repair are not conducted during the entire service life, there is only one continuous interval, namely $0 \leq t \leq t_D$. The probability of failure, P_o, in this time interval is given as follows.

$$P_o = P[t \leq t_D, \quad k \geq k_c] \tag{6-29}$$

Probability of Failure Under One Inspection

The design life is again assumed to be t_D hours. An inspection is assumed to be performed for fatigue damage at time $t = t_o = t_D/2$ hours. During this inspection, if a

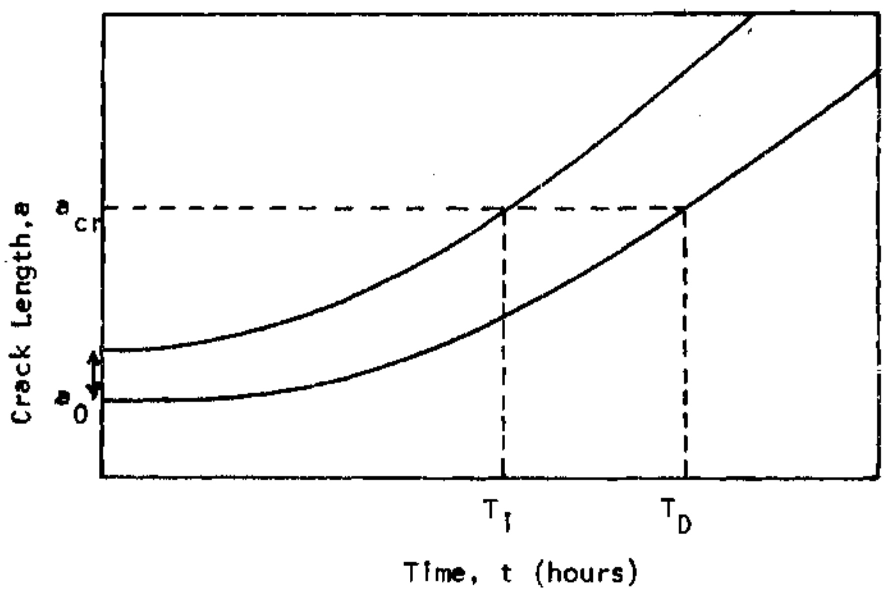

Figure 6-1a. Fail Safety by Inspection

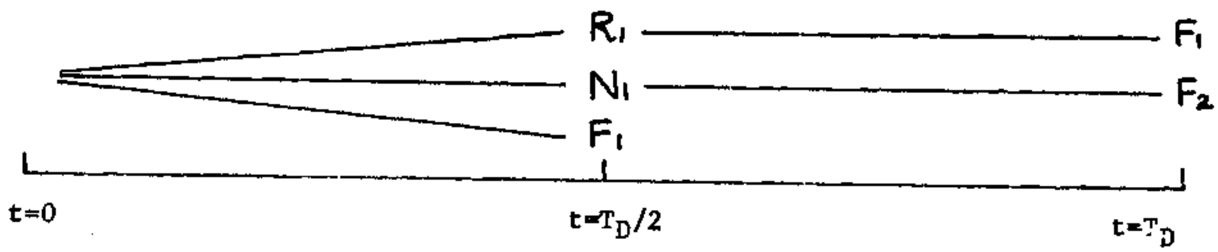

Figure 6-1b. One Inspection: Possible Events

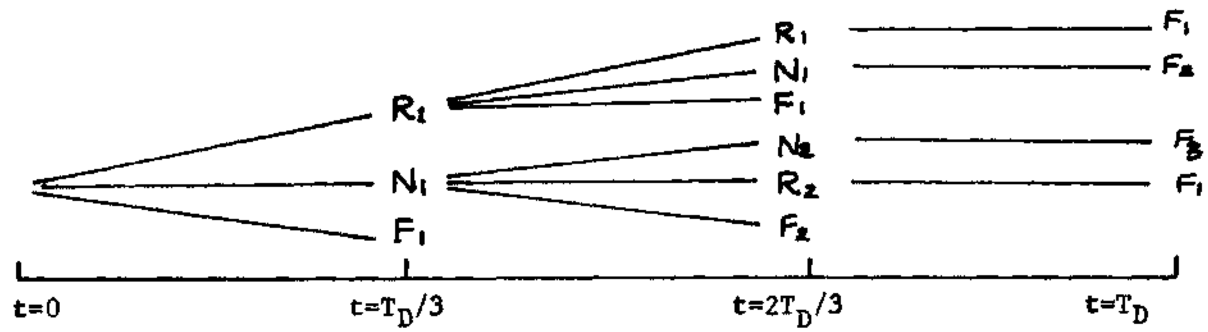

Figure 6-2. Two Inspections: Possible Events

crack of size $k_R \leq k < k_c$ is observed, it is repaired immediately. After the repair is done, the structure is assumed to be as good as new. If the crack length is less than k_R it is not repaired. The determination of the probability of failure under one inspection requires the consideration of all possible events of failure. Failure can take place in three mutually exclusive and exhaustive events.

First, failure of some of the aircraft can take place in the time interval $0 \leq t \leq t_o$. The probability of this event is given by the following.

$$F_1 = \sum_{k=k_c}^{N_{max}} P[t_o,k] \tag{6-29a}$$

Secondly, some of the survived aircraft are repaired and can fail in the time interval $t_o \leq t \leq t_D$. Since the structure is assumed to be as good as new, the time is counted from zero once again for the repaired fraction of the aircraft fleet. With this assumption, the probability of this compound event is given by the following.

$$P_a = [\sum_{k=k_R}^{k_c-1} P(t_o,k)D(k)][\sum_{k=k_c}^{N_{max}} P(t_o,k)] \tag{6-29b}$$

This expression includes the detection probability

D(k) when a crack of length $k\Delta\ell$ is present. The terms in the first square brackets give the repair probability at time t_o. The second square bracket contains terms giving the probability of failure in the interval $0 \le t \le t_o$.

Thirdly, the remaining fleet with crack size of $0 \le k \le k_R$ which are not repaired, can fail in the time interval $t_o \le t \le t_D$. The probability of no-repair at time t_o is given as follows

$$N_1 = \sum_{k=0}^{k_R-1} P[t_o,k] \tag{6-29c}$$

The unconditional probability of failure in the interval $t_o \le t \le t_D$ is given by the expression

$$P_b = \sum_{k=k_c}^{N_{max}} P[t_D,k] - \sum_{k_a-k_c}^{N_{max}} P[t_o,k] \tag{6-29d}$$

Then the conditional probability of failure in the same interval given the condition that the failure did not take place in the first interval, i.e., $0 \le t \le t_o$ is obtained by normalizing the previous expression by the following expression

$$P_c = 1 - \sum_{k-k_c}^{N_{max}} P[t_o,k] \tag{6-29e}$$

The probability of this third event is now written as

$$P_d = \left[\sum_{k=0}^{k_R-1} P[t_o,k]\right] \frac{\left\{\sum_{k=k_c}^{N_{max}} P[t_D,k] - \sum_{k=k_c}^{N_{max}} P[t_o,k]\right\}}{1 - \sum_{k=k_c}^{N_{max}} P[t_o,k]} \qquad (6\text{-}29f)$$

The expressions for various probabilities can be simplified by introducing a convenient notation as follows:

<u>Notation</u>:

Let F_1 = Prob. of failure at the end of the 1st interval

$$= \sum_{k=k_c}^{N_{max}} P[t_o,k] \qquad (6\text{-}30a)$$

F_2 = Prob. of failure at the end of the 2nd interval

$$= \sum_{k=k_c}^{N_{max}} P[t_o,k] \qquad (6\text{-}31)$$

R_1 = Prob. of repair at the end of the 1st interval

$$= \sum_{k=k_R}^{k_c-1} D(k)P[t_D,k] \qquad (6\text{-}32)$$

N_1 = Prob. of no repair at the end of the 1st interval

$$= \sum_{k=k_R}^{k_c-1} [1-D(k)]P[t_o,k] + \sum_{k=0}^{k_R-1} P[t_o,k] \qquad (6\text{-}33)$$

It is interesting to note that

$$F_1 + R_1 + N_1 = 1.0 \tag{6-34}$$

Following this notation, the probability of failure in the design life of t_D hours under one inspection at t_o hours is the following:

$$P1 = F_1 + R_1F_1 + N_1 \frac{(F_2 - F_1)}{(1 - F_1)} \tag{6-35}$$

This expression can be checked for various special cases as follows:

(1) No inspection or repair: i.e., $R_1 = 0$

If at time $t = t_o$ hours no inspection is done, then $R_1 = 0$ in equation (6-34). In such a case it yields the following.

$$F1 + N1 = 1$$

(6-36a)

or $$N_1 = 1 - F1$$

Then,

$$P_1 = F_1 + (1 - F_1) \frac{(F_2 - F_1)}{(1 - F_1)}$$

(6-36)

$$= F_1 + F_2 - F_1$$

i.e.,

$$P_1 = F_2$$

This is true because, when the inspection at time $t = t_o$ hours is eliminated, there is only one long interval, $0 \leq t \leq t_D$. The probability of failure in this interval is given by

$$P_e = \sum_{k=k_c}^{N_{max}} P(t_D,k) \qquad (6\text{-}36a)$$

which, by the assumed notation is equal to F_2.

(2) Certain Repair: i.e. $N_1 = 0$

Part of the fleet failed in the 1st interval and the rest of the fleet is repaired. For this special case

$$F_1 + R_1 = 1, \quad \text{or} \quad R_1 = 1-F_1 \qquad (6\text{-}37a)$$

Then, the original expression for P_1 reduces to the following.

$$P_1 = F_1 + (1-F_1)F_1 \qquad (6\text{-}37)$$

(3) No failure in the first interval, i.e. $F_1 = 0$

None of the fleet failed in the first interval, some are repaired, and the rest are not repaired. For this special case

$$R_1 + N_1 = 1, \quad \text{or} \quad N_1 = 1-R_1 \tag{6-38a}$$

Then, the original expression for P_1 simplifies to

$$P_1 = (1-R_1)F_2 \tag{6-38}$$

Notation:

Prob. of failure under j inspections = P_j

Prob. of failure at the ith inspection time, $iT_o = F_i$

Prob. of repair at the ith inspection time, $iT_o = R_i$

Prob. of no-repair at the ith inspection time, $iT_o = N_i$

$$\text{Note that } F_i + R_i + N_i = 1 \tag{6-39}$$

Probability of Failure under Two Inspections

The design life is t_D hours. Two intermediate inspections are conducted at time intervals of $t_o = t_D/3$. The criteria for repair, no-repair, and failure remain the same as before, under one inspection.

All the possible events of failure under two inspections can be schematically represented as follows. The derivation for the probability of failure under two inspections results in the following expression:

$$P_2 = F_1 + R_1F_1 + N_1 \left(\frac{F_2-F_1}{1-F_1}\right)$$

$$+ R_1R_1F_1 + R_1N_1 \left(\frac{F_2-F_1}{1-F_1}\right)$$

$$+ \frac{N_1N_2}{(1-F_1)} \left(\frac{F_3-F_2}{1-F_2}\right) + \frac{N_1R_2F_1}{(1-F_1)} \tag{6-40}$$

Check: The expression for P_2 can be checked for special cases as follows

<u>Case 1</u>: <u>No repairs at $t_D/3$ and $2t_D/3$</u>

For this case $R_1 = R_2 = 0$

This implies that

$$F_1 + N_1 = 1$$
$$F_2 + N_2 = 1 \tag{6-41}$$

Substituting these expressions in the expression for P_2, it follows that

$$P_2 = F_1 + (1-F_1)\frac{(F_2-F_1)}{(1-F_1)} + \frac{(1-F_1)(1-F_2)}{(1-F_1)}\frac{(F_3-F_2)}{(1-F_2)}$$

$$= F_1 + (F_2-F_1) + (F_3-F_2)$$

i.e. $$P_2 = F_3 \tag{6-42}$$

Case ii: Certain Repairs

For this case $N_1 = N_2 = 0$

This implies that

$$F_1 + R_1 = 1$$
$$F_2 + R_2 = 1 \tag{6-43}$$

Substituting these in the expression for P_2,

$$P_2 = F_1 + (1-F_1)F_1 + (1-F_1)^2F_1$$
$$= F_1[1+(1-F_1) + (1-F_1)^2] \tag{6-44}$$

Case iii: No failure in the 1st two intervals

For this case, $F_1 = F_2 = 0$

This implies that

$$R_1 + N_1 = 1$$
$$R_2 + N_2 = 1 \tag{6-45}$$

Substituting these in the expression for P_2,

$$P_2 = (1-R_1)(1-R_2)F_3 \tag{6-46}$$

Probability of Failure Under 'j' Inspections

Now the expressions are generalized to obtain the probability of failure when 'j' inspections are performed during the design life of the structure. The design life is assumed to be t_D hours. The number of intermediate inspections is 'j', at time intervals of t_D/j hours.

The general expression can be written by induction as follows:

$$P_j = \sum_{i=o}^{j} (R + \frac{N}{1-F})^i F_1 \tag{6-47}$$

This equation is not an algebraic one. The quantity in parentheses represents operations, and 'i' stands for the number of operations.

Check:

$$P_o = F_1$$

$$P_1 = F_1 + R_1F_1 + \frac{N_1}{1-F_1} (F_2-F_1)$$

$$P_2 = F_1 + R_1F_1 + \frac{N_1}{1-F_1} (F_2-F_1) + R_1R_1F_1 +$$

$$\frac{R_1N_1}{(1-F_1)} (F_2-F_1) + \frac{N_1N_2}{(1-F_1)} (\frac{F_3-F_2}{1-F_2}) + \frac{N_1R_2F_1}{(1-F_1)} \tag{6-48}$$

Similar reasoning with a very similar procedure is used for estimating the effect of inspections on the probability of mode E_s fatigue failure. Then, the probability of failure for the stiffened panel is known completely and equation (6-28) can be determined. The application of all the 'tools' developed so far is demonstrated by a numerical example in the next chapter.

CHAPTER VII

ILLUSTRATIVE EXAMPLE I

7-1. Problem Description

The application of the various conceptual tools developed in the previous chapters is demonstrated in this chapter by means of an illustrative example. Currently, the reliability based fatigue design of a built-up structure such as a sheet-stiffener combination is undertaken. Figure 3-1 illustrates the stiffened panel configuration. The panel is of width w and thickness t. The panel is to be made from one of several available materials. The particular assembly is assumed to be a subassembly of an aircraft structure. The external loading F consists of a sustained loading F_1 and a random fatigue loading F_2. It is assumed that the random fatigue loading F_2 is specified probabilistically. On the panel there are stringers attached at a regular spacing of 2b. The fatigue crack propagation is a function of this spacing also. The more the spacing is the faster the crack grows and vice versa. The total width w is assumed to be fixed and t and b are the only geometric design variables to be determined. The panel is subjected to regular maintenance inspections periodically during the expected design life span. The number of inspections j is the maintenance design variable. Finally, there are the material variables corresponding to each of the available materials.

For a given panel configuration the initial (residual) strength i.e. when the crack length is zero, is denoted σ_R. The external applied stress corresponding to the external loading is denoted σ_L. If σ_R and σ_L are deterministic, the initial safety margin i.e. before fatigue crack initiation, is given by the ratio of σ_R to σ_L. As explained earlier, however, both σ_R and σ_L have uncertainties and have probabilistic descriptions. Then, the initial reliability is considered as a safety measure. This is represented by the probability that (σ_R/σ_L) is greater than unity. Due to the presence of fatigue loading crack growth takes place. The presence of a crack of length a_i reduces the initial strength σ_R to a lower residual strength σ_{Ri}. Because of the lower residual strength the reliability also decreases. The probability of failure increases as the crack length increases to such an extent (a_{cr}) that the strength is reduced below the externally applied stress σ_L. The probability of failure can be reduced by increasing the thickness and/or number of stringers and/or number of inspections. However this would increase the total weight/cost of the panel structure. Therefore, the design procedure consists of selecting the design variables, i.e. t, b, j and the material so as to minimize the total expected cost/weight. This entire procedure however is subject to the restraint that the margin of safety or reliability does not fall below an acceptable limit during the projected life of the structure.

7-2. Objective Function for Optimization

For aircraft structures weight is the most crucial consideration. However, the weight can be reflected on monetary costs. In the present context the total cost of the stiffened panel is considered to be minimized. The total cost function comprises of three components as explained in equation (3-1). Firstly, there is the "expected cost of failure" which is the product of the cost of failure and the total probability of failure. The cost of failure is much higher than the purchasing cost of the panel because of the extraneous expenditure due to failure such as loss of equipment, life, higher insurance etc. The second component cost is the deterministic production cost of the panel. This is a function of the panel thickness, stringer spacing and the material. The last component cost is the cost of inspections for fatigue damage which are scheduled to be conducted periodically during the lapse of the design life. This is the product of the number of inspections and the cost of one inspection. The mathematical form of the total cost function is then given as follows.

$$C_T' = (K_1 P_f) + (K_2 t + K_3 N_{st}) + (K_4 j) \tag{7-1}$$

where

K_1 = cost of failure

K_2 = cost per unit thickness

K_3 = cost per stringer

K_4 = cost of an inspection

Rearranging equation (7-1) by dividing throughout by the cost of failure K_1, it reduces to the following.

$$C_T = P_f + C_2 t + C_3 N_{st} + C_4 j \qquad (7\text{-}2)$$

where

$C_T = C_T'/K_1$

$C_2 = K_2/K_1$

$C_3 = K_3/K_1$

$C_4 = K_4/K_1$

The coefficients C_2, C_3 and C_4 vary for different materials. Also, in equation (7-2) t is the thickness of the panel, N_{st} is the number of stringers, j is the number of inspections and P_f is the total probability of failure.

7-3. Solution Methodology

The following are the essential steps to be followed in the methodology for the reliability-based fail-safe fatigue design procedure.

(1) From the known initial flaw distribution, the parameters of the stochastic model for the crack growth

(equations (4-42) and (4-43)) are determined by numerical techniques discussed earlier.

(2) The residual strength σ_R is obtained as a function of the crack length a_{cr} with the stringer spacing 2b as a parameter by means of equation (6-4).

(3) The static reliability R_s is obtained as a function of the thickness and residual strength by following equations (6-51) and (6-54).

(4) The probability of dynamic fatigue failure under j inspections P_j is obtained as a function of the residual strength/critical crack length by means of equation (6-73).

(5) The total probability of failure is obtained as a function of the probability of fatigue failure in which the probability of static failure is a parameter, from equation (6-30).

(6) Then a set of trial design variables are selected, by using the reliability constraint as a guidance and the total probability of failure is calculated from step (5).

(7) The trial design variables and the total probability of failure are substituted into the total expected weight/cost function.

(8) The minimization is carried out by means of search method. For fixed values of j, R_s and b the total cost function is calculated as a function of the thickness for each material.

(9) The above steps are repeated for each of the several available materials and the combination of the design variables, giving the lowest total cost is chosen as the optimum values.

7-4. Numerical Example: Results and Conclusions

In order to illustrate the developed method, the optimum design of the stiffened panel is carried out for the following situation.

(a) Material has to be chosen from among three aluminum alloys designated as (i) 7075-T6, (ii) 2024-T3, (iii) 2024-T6.

(b) Geometric variables t and N_{st} are not restricted

(c) The inspection variable j is not restricted.

In addition, the data given in Table 7-1 for each material is assumed to be given. As a first step, the residual strength-critical crack length diagrams are obtained with number of stringers as a parameter, as shown in Figure 7-1. As the number of stringers increases the stringer spacing decreases and the rate of crack growth decreases. The tip stress reduction factor C_R (a/b) required to calculate the residual strength is obtained from reference [24] as shown in Figure 7-2. The variation of the static reliability R_s with residual strength and thickness is shown in Figure 7-3. For a given loading spectrum, in order to maintain the same static reliability, a larger thickness is required to lower

Table 7-1. Given Data

Material 1	Material 2	Material 3
α = 8.9	8.4	8.1
β = 5000 cycles	5500 cycles	6000 cycles
$\mu_{\Delta L}$ = 1000.0 lbs/in	1000.0 lbs/in	1000.0 lbs/in
$\sigma^2_{\Delta L}$ = 100.0 lbs^2/in^2	100.0 lbs^2/in^2	100.0 lbs^2/in^2
μ_r = 0.5	0.5	0.5
σ^2_r = 0.01	0.01	0.01
$\rho_{\Delta L,r}$ = 0.01	0.01	0.01
K_{ic} = 6800 $lbs/in^{3/2}$	85000 $lbs/in^{3/2}$	88000 $lbs/in^{3/2}$
n = 3	3	3
C_1 = $5x10^{-13}$	$4.5x10^{-13}$	$3.5x10^{-13}$
K_2 = $3.0x10^{-5}$	$4.0x10^{-5}$	$5.0x10^{-5}$
K_3 = $9.0x10^{-7}$	$16.0x10^{-7}$	$25.0x10^{-7}$
K_4 = $12.0x10^{-7}$	$20.0x10^{-7}$	$30.0x10^{-7}$

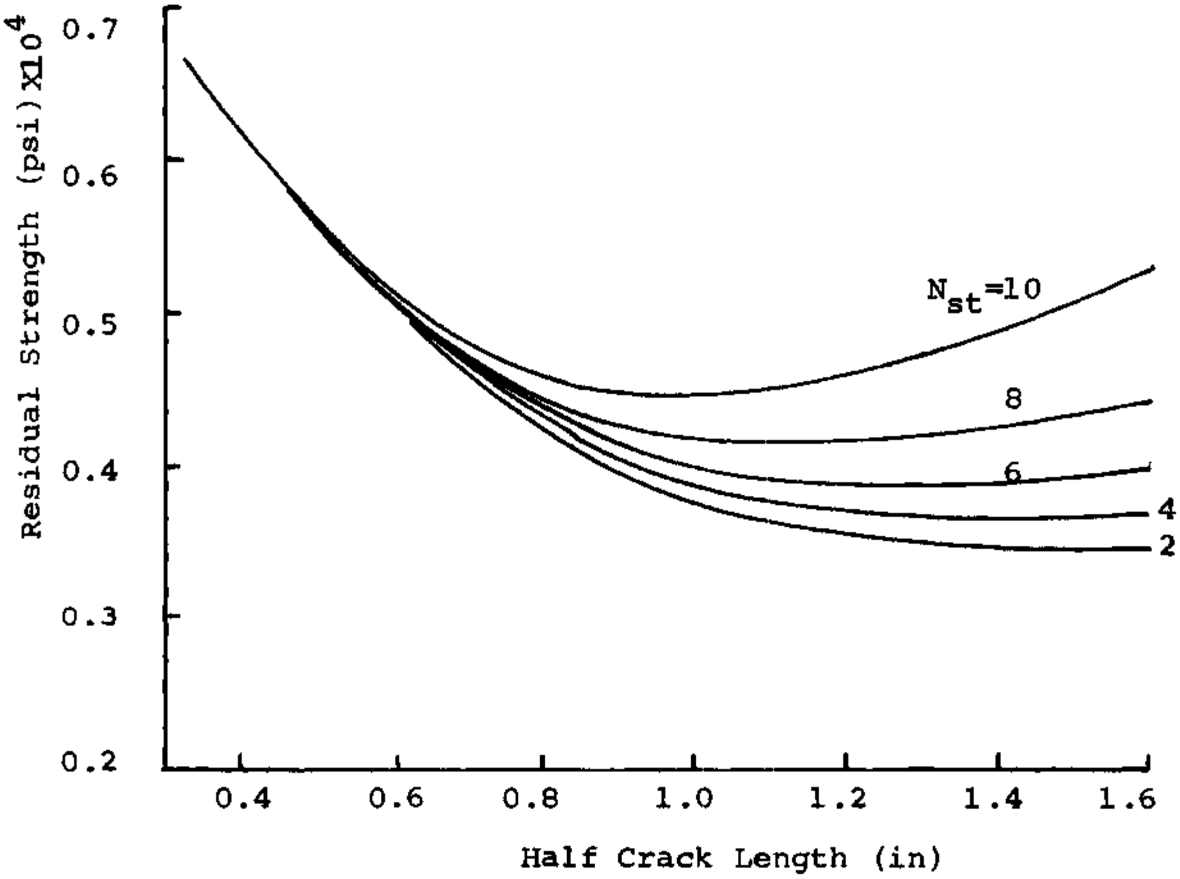

Figure 7-1. Residual Strength (psi)

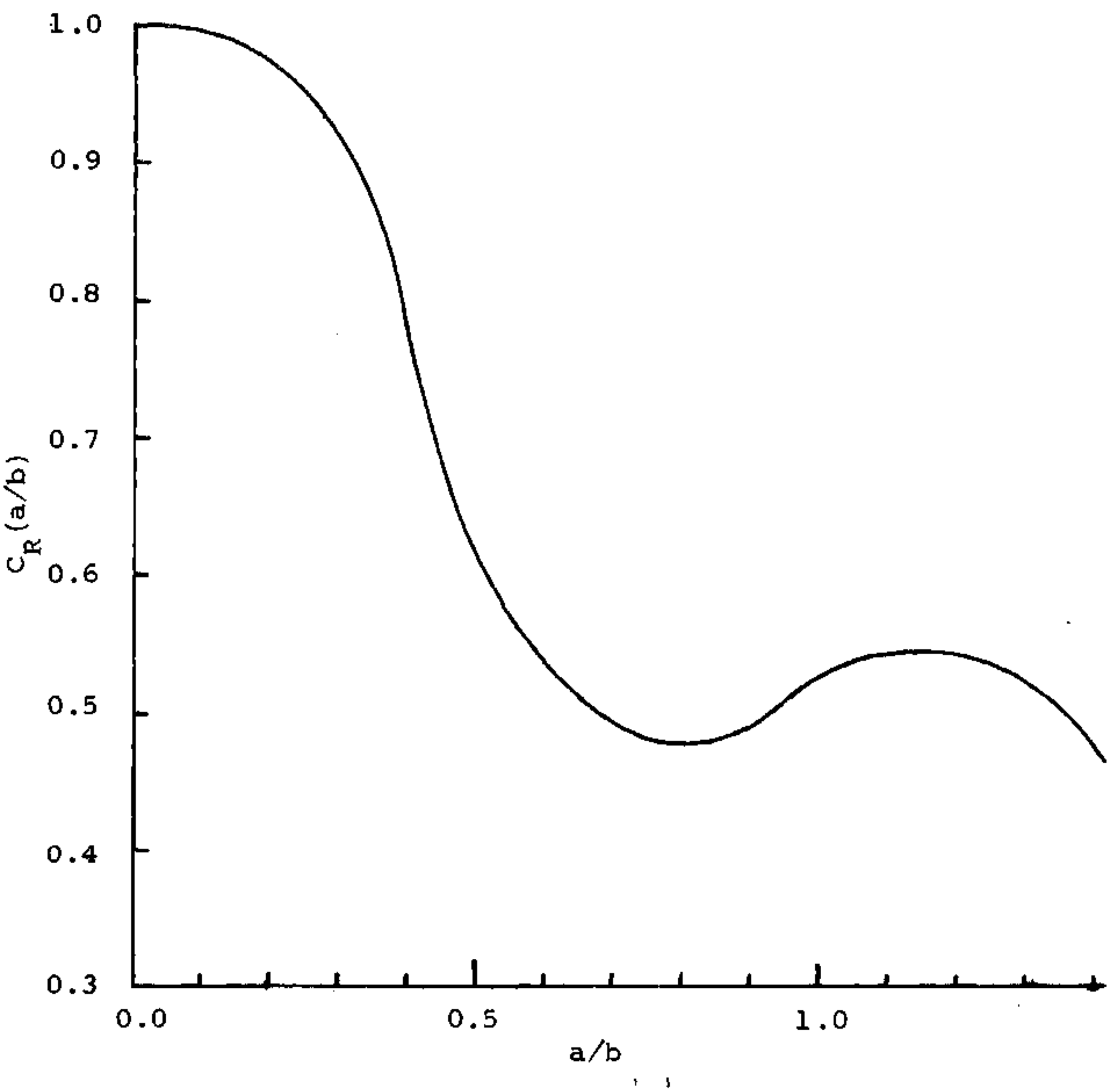

Figure 7-2. Tip Stress Reduction Factor, $C_R(a/b)$

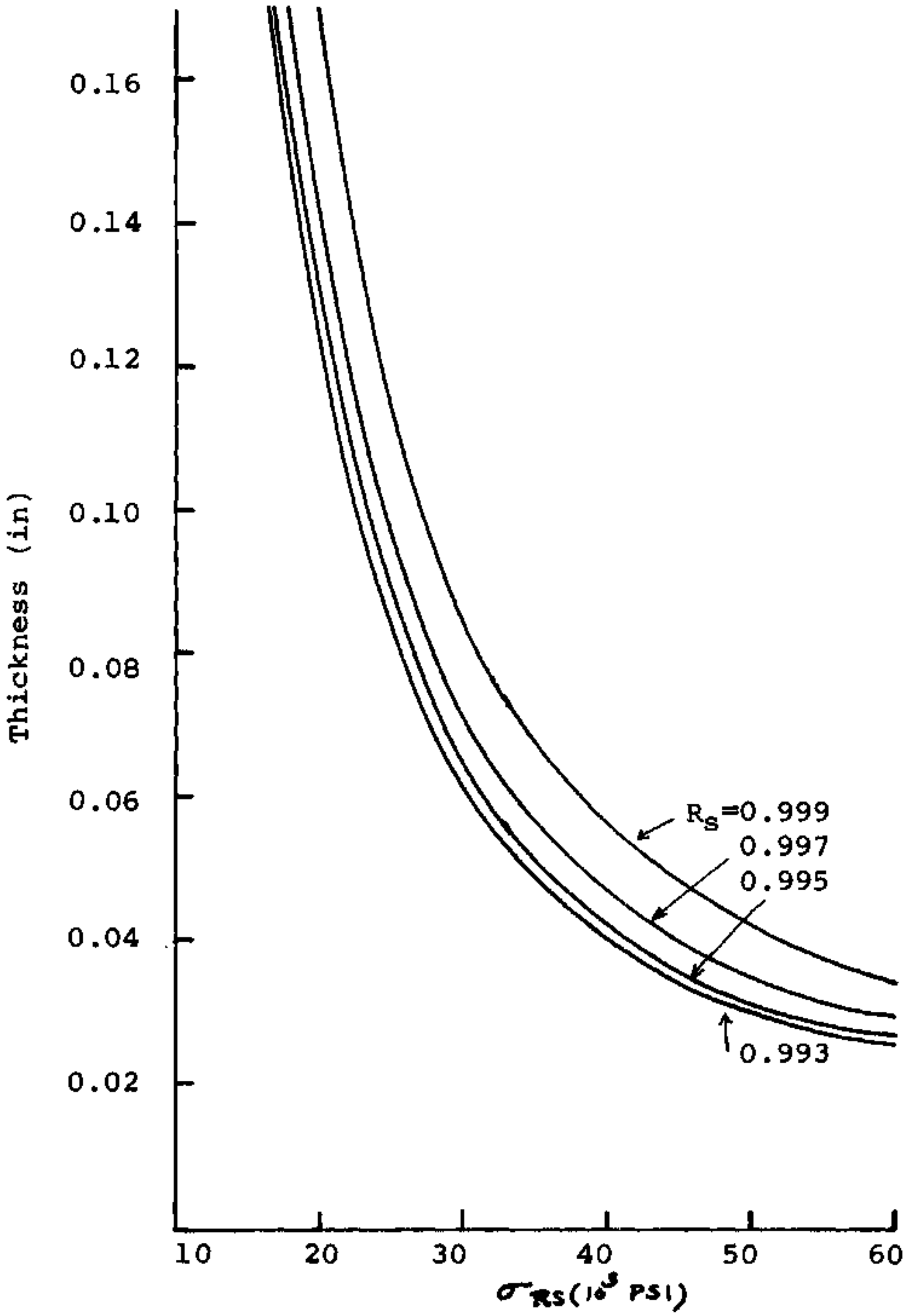

Figure 7-3. Static Reliability vs. Thickness

the design residual stress. In Figure 7-4 the relation between the probabilities of static failure, dynamic failure and the total probability of failure is delineated.

7-5. Results and Conclusions

Figure 7-5 presents the minimization curves for $j = 4$ and $N_{st} = 2$, 4 and 6. From these curves it is obvious that the minimum value occurs at a lower thickness as the number of stringers increases, while the probability of failure is decreasing. For example, the minimum for two stringers is 0.21007×10^{-4} at a thickness of 0.258" with the corresponding probability of failure of 0.7271×10^{-5}. For four stringers the minimum is 0.19007×10^{-4} at a thickness of 0.179" and the probability of failure is 0.5239×10^{-5}. Similar by for six stringers the minimum is 0.1946×10^{-4} at a thickness of 0.1475" with a probability of failure of 0.4241×10^{-5}. Thus the overall minimum for four inspections from Figure 7-5 is for four stringers and a thickness of 0.179".

Figure 7-6 depicts the minimization curves for material 1 for five inspections and various number of s stringers. The overall minimum from Figure 7-6 is 0.1879×10^{-4} at 0.1632" and four stringers. Similarly Figures 7-7 and 7-8 delineate the similar curves for six and seven inspections,,respectively. It is obvious from Figures 7-5 through 7-8 that the minimum most is 0.1879×10^{-4} at 0.1632"

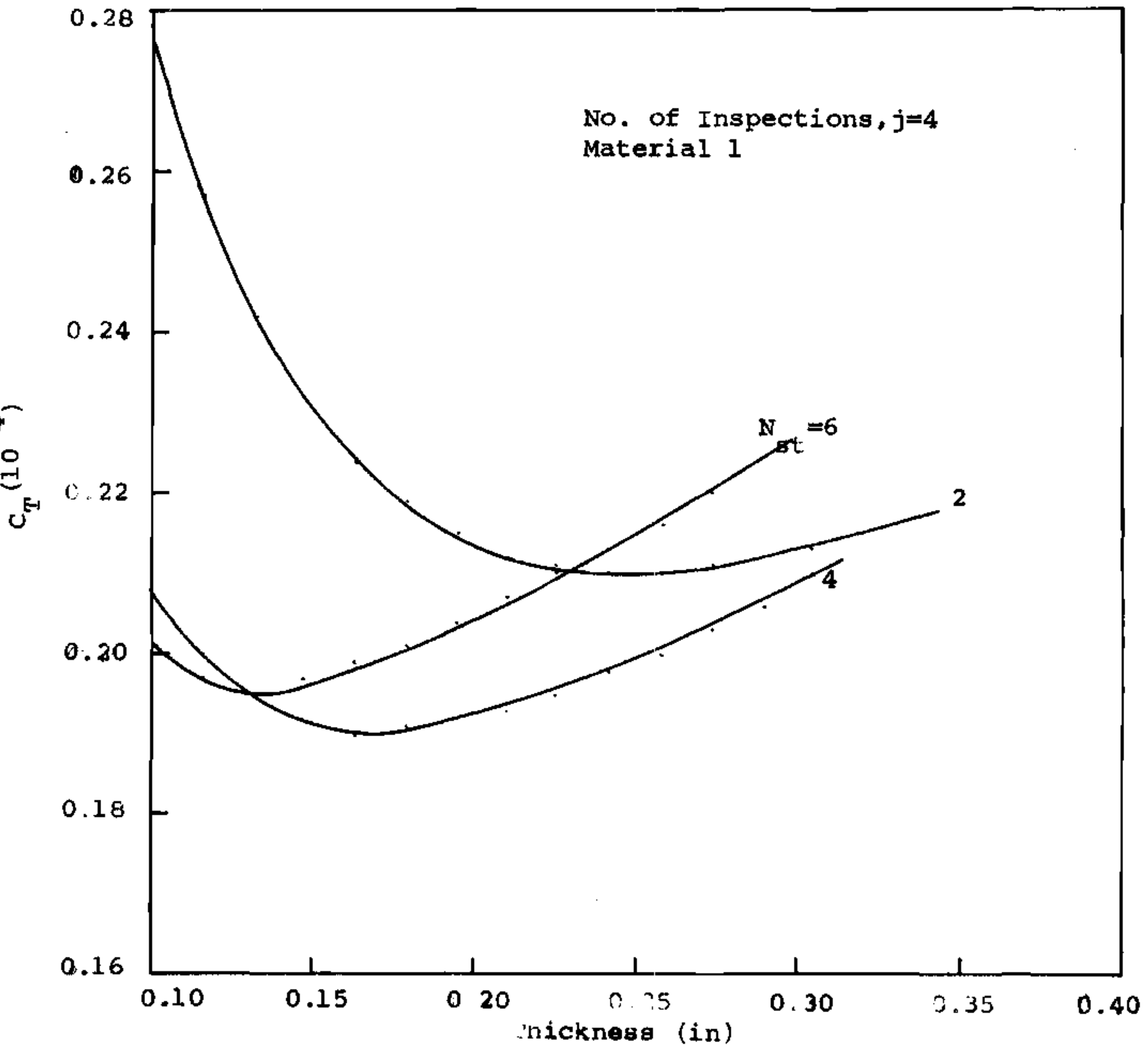

Figure 7-4. Minimization Curves

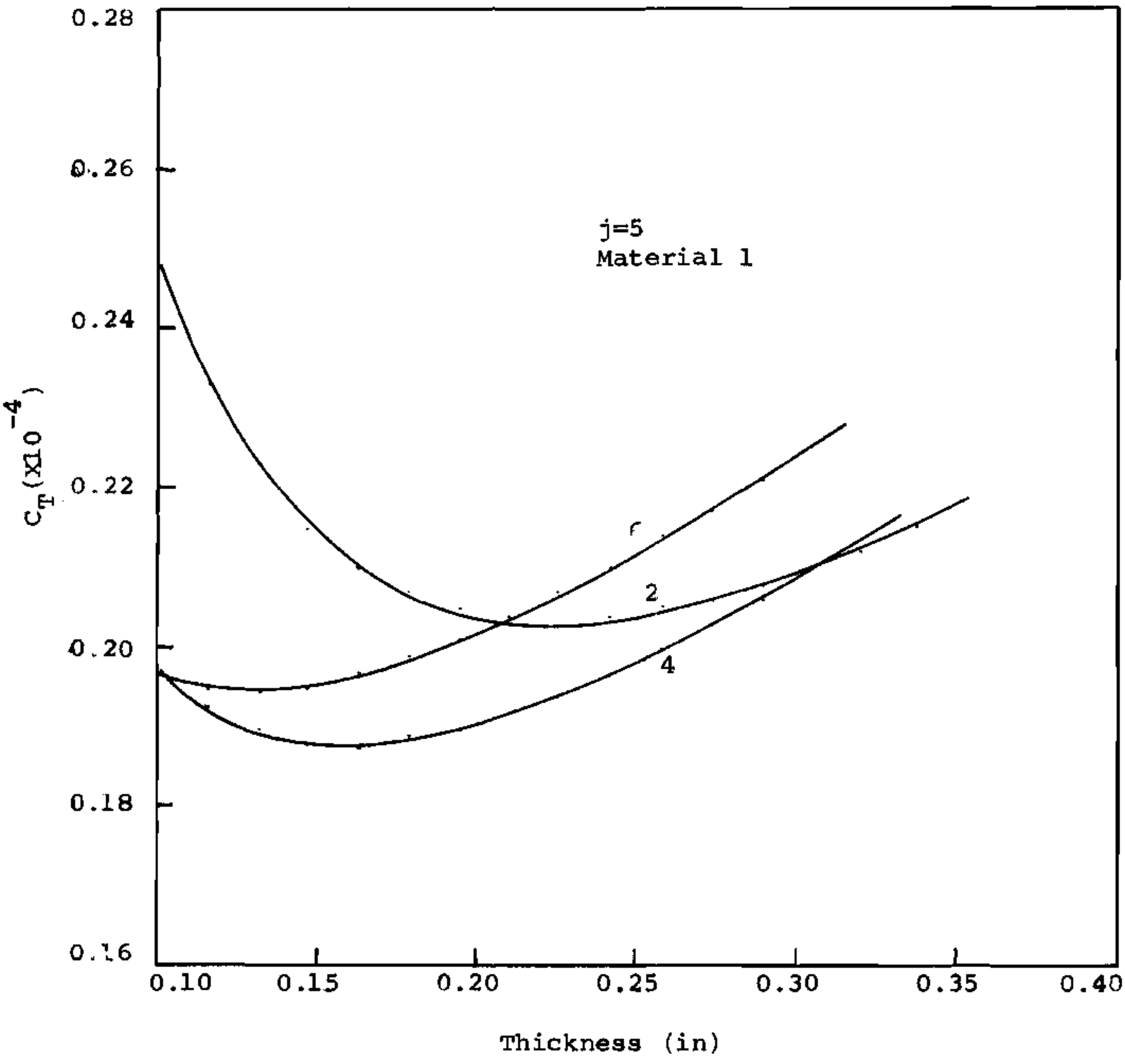

Figure 7-5. Minimization Curves

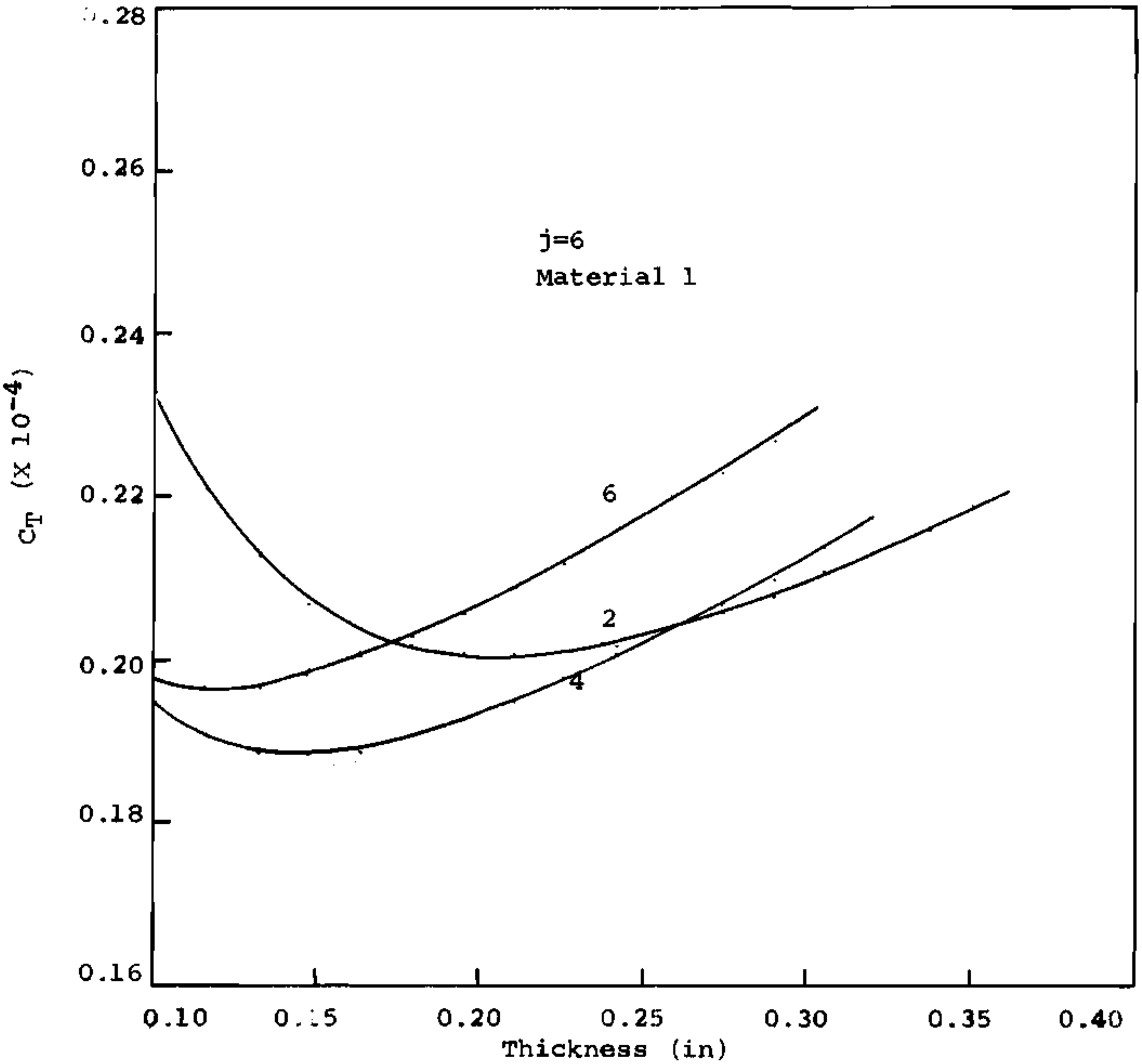

Figure 7-6. Minimization Curves

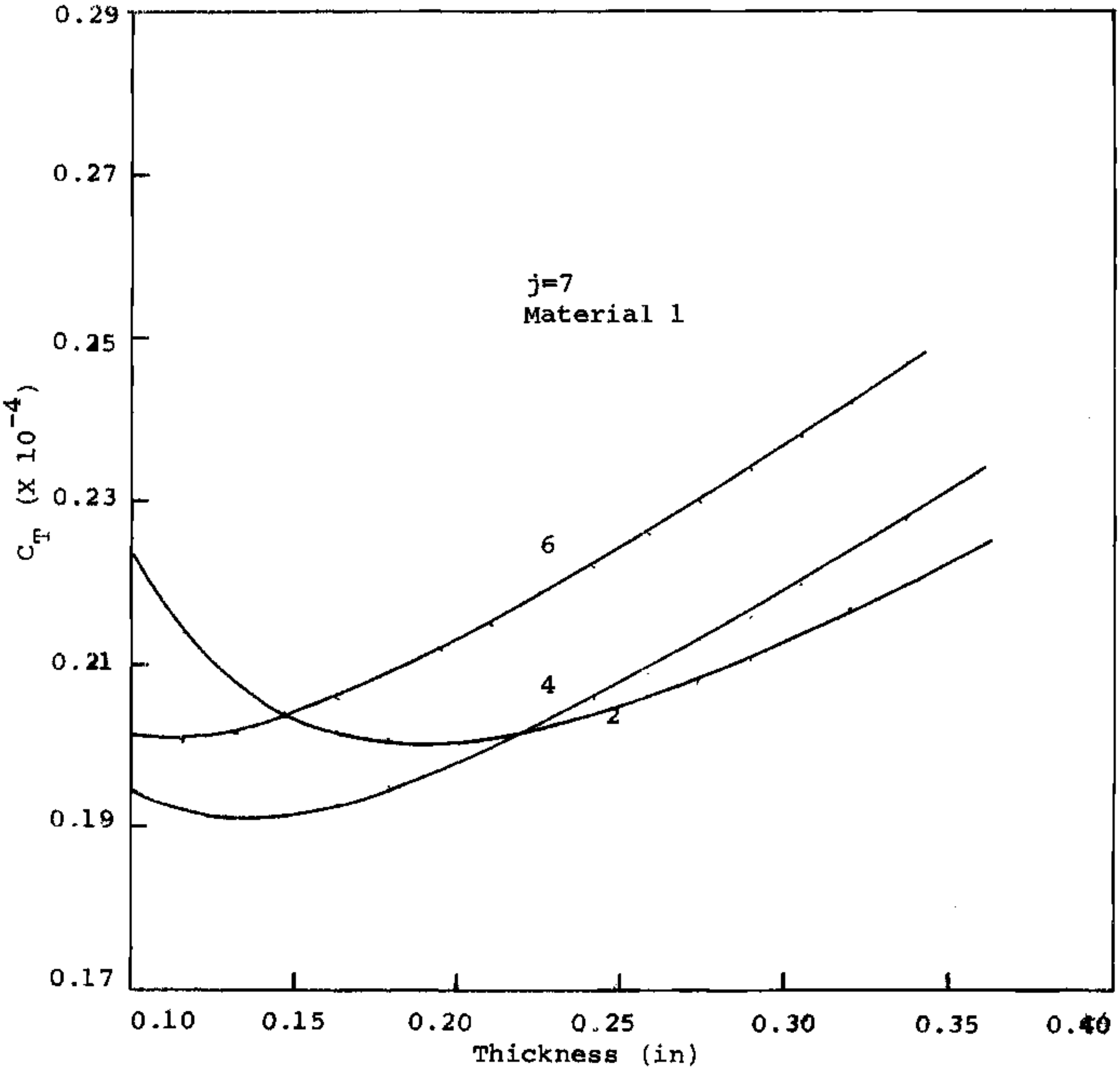

Figure 7-7. Minimization Curves

and four stringers. The probability of failure corresponding to this case is 0.4596×10^{-5}.

Next, Figures 7-9 through 7-12 are drawn for material 2. The minimum most from these graphs is 0.2014×10^{-4} at a thickness of 0.179", four stringers and five inspections. The corresponding probability of failure is 0.6985×10^{-5}. The last set of figures, 7-13 through 7-16 are for material 3. From these the minimum most is 0.42917×10^{-4} at a thickness of 0.2105", four stringers and four inspections. The probability of failure is 0.1039×10^{-4}. Now comparing the minimum most values for the three materials, it is obvious that material 1 gives the least total cost. Thus the choice material is material 1 which is 7075-T6 aluminum alloy.

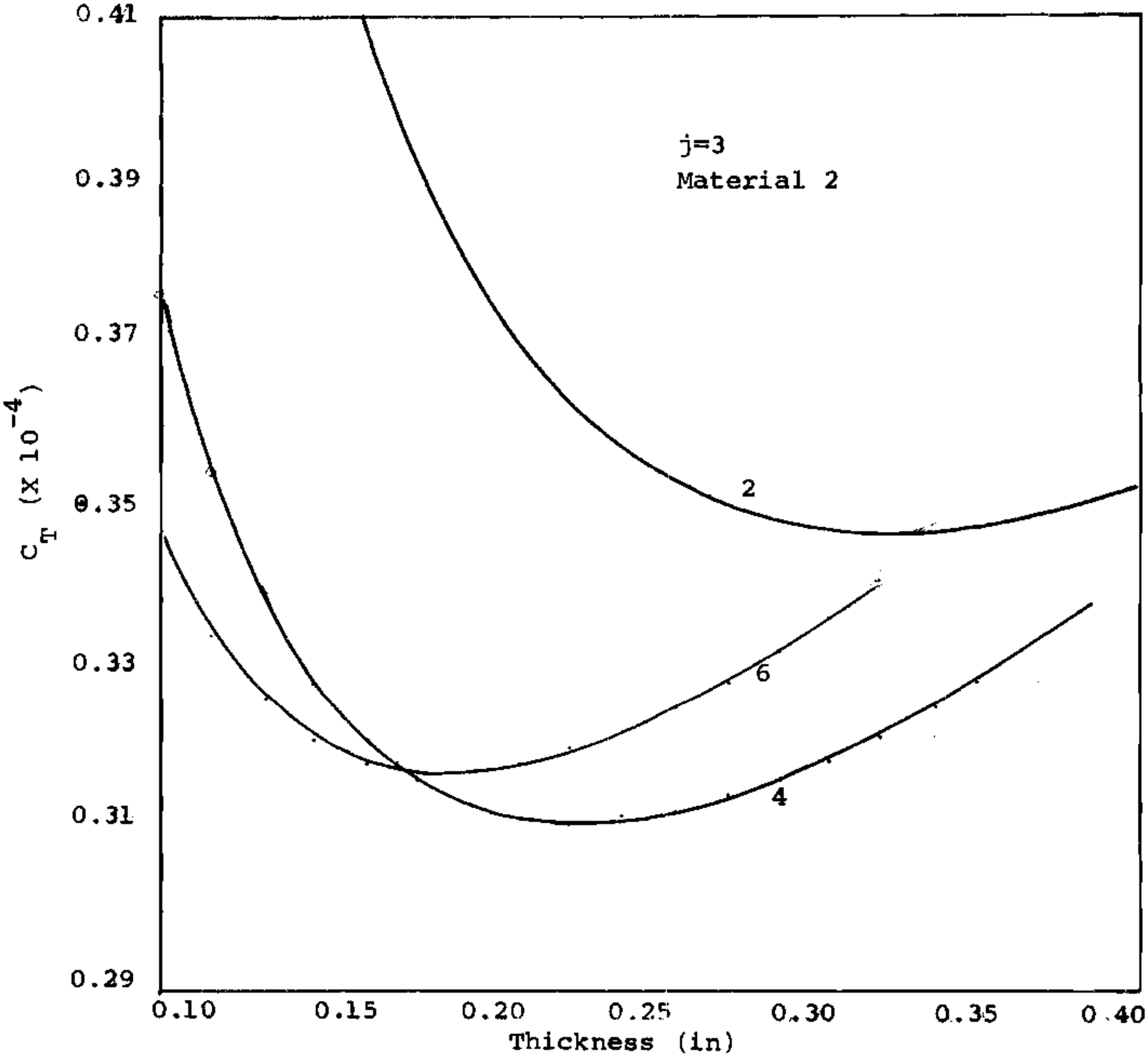

Figure 7-8. Minimization Curves

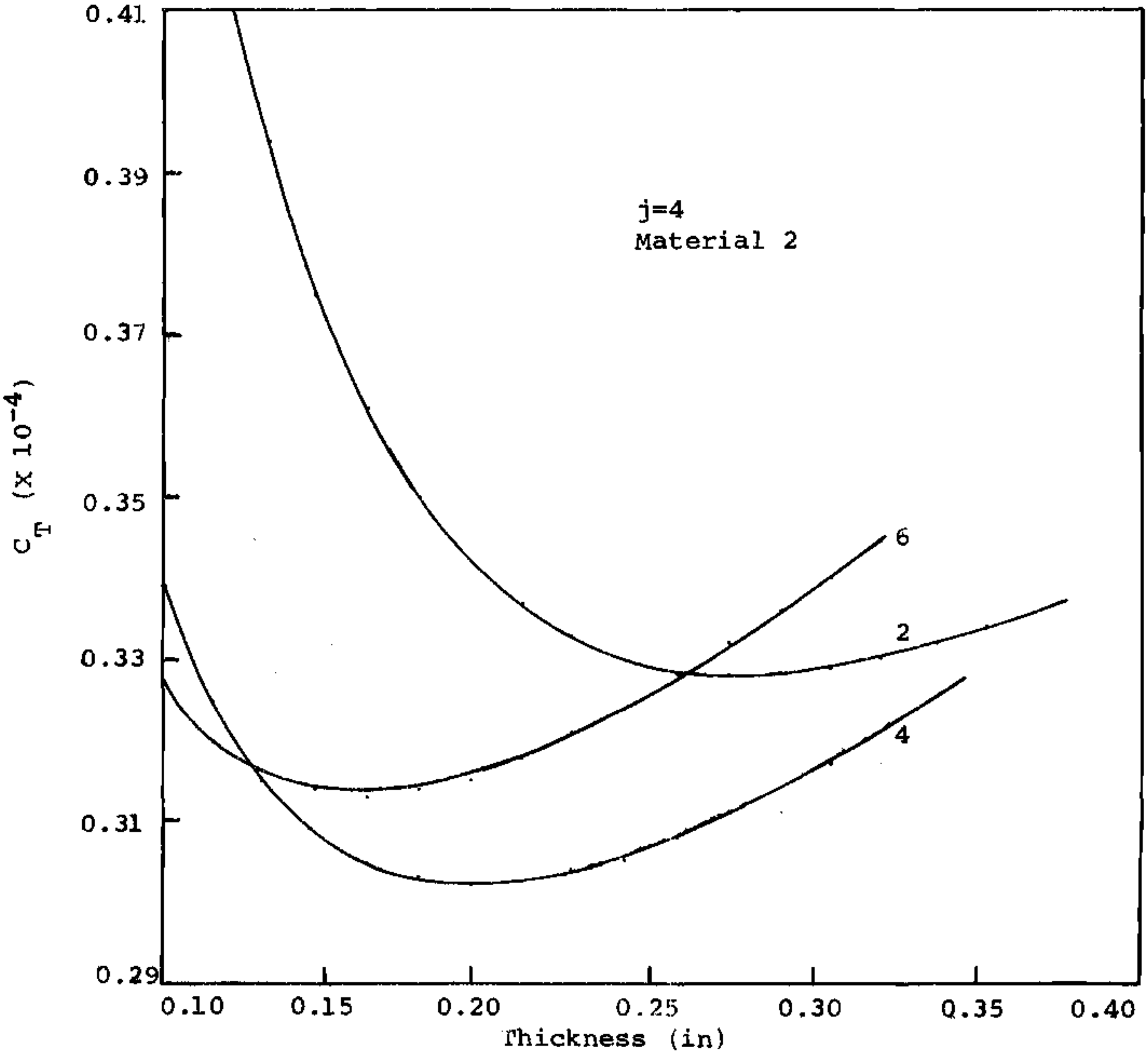

Figure 7-9. Minimization Curves

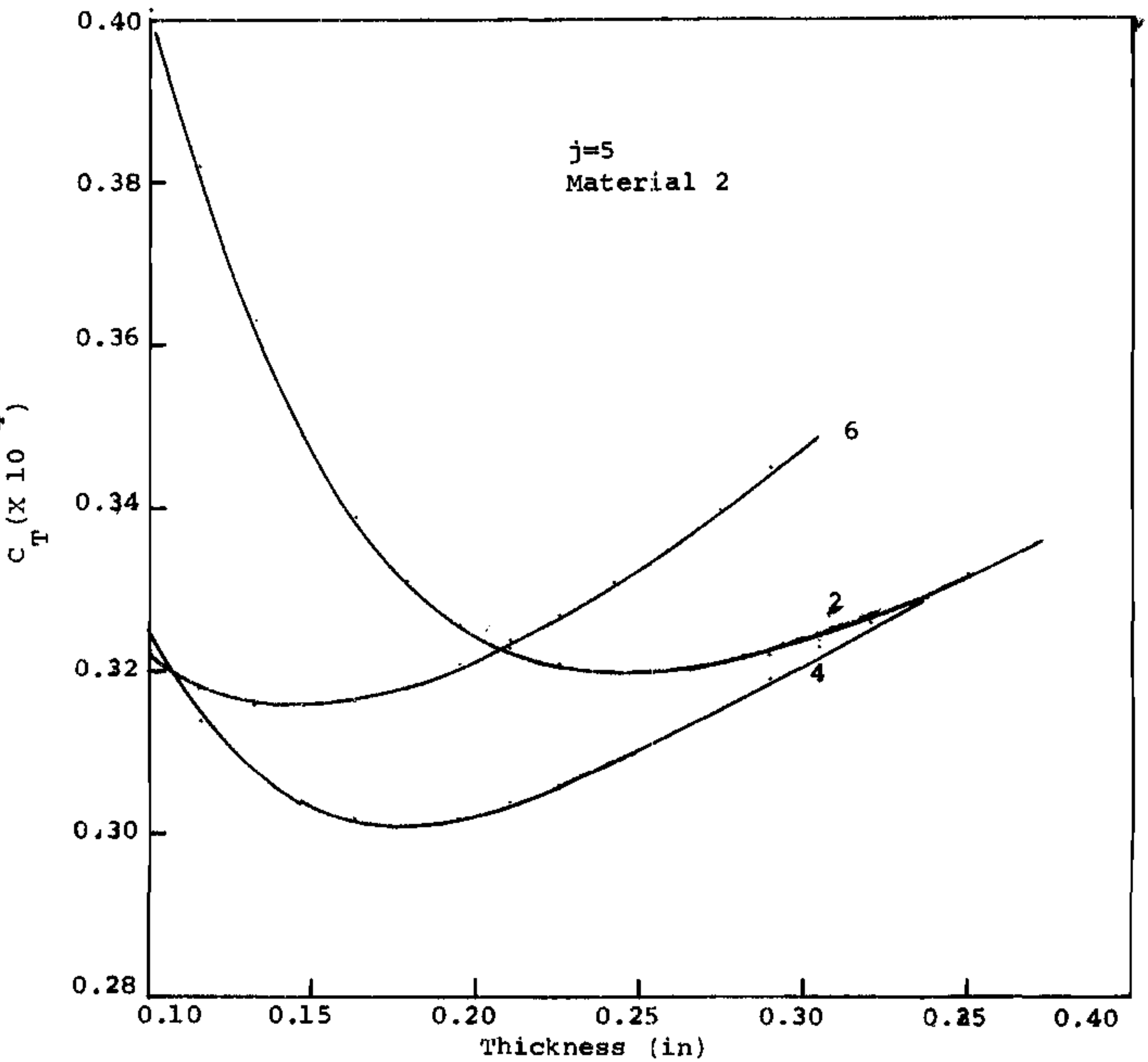

Figure 7-10. Minimization Curves

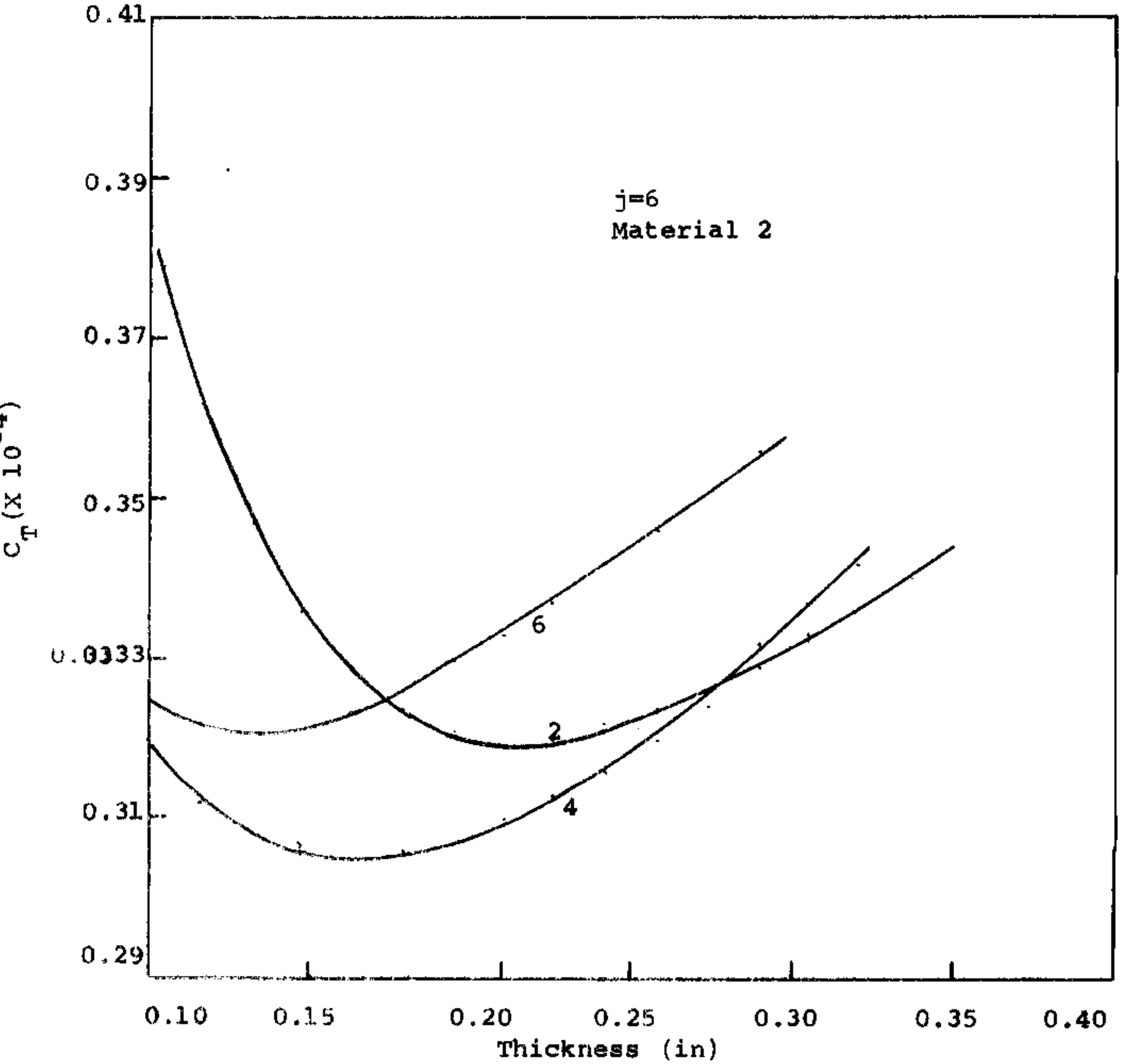

Figure 7-11. Minimization Curves

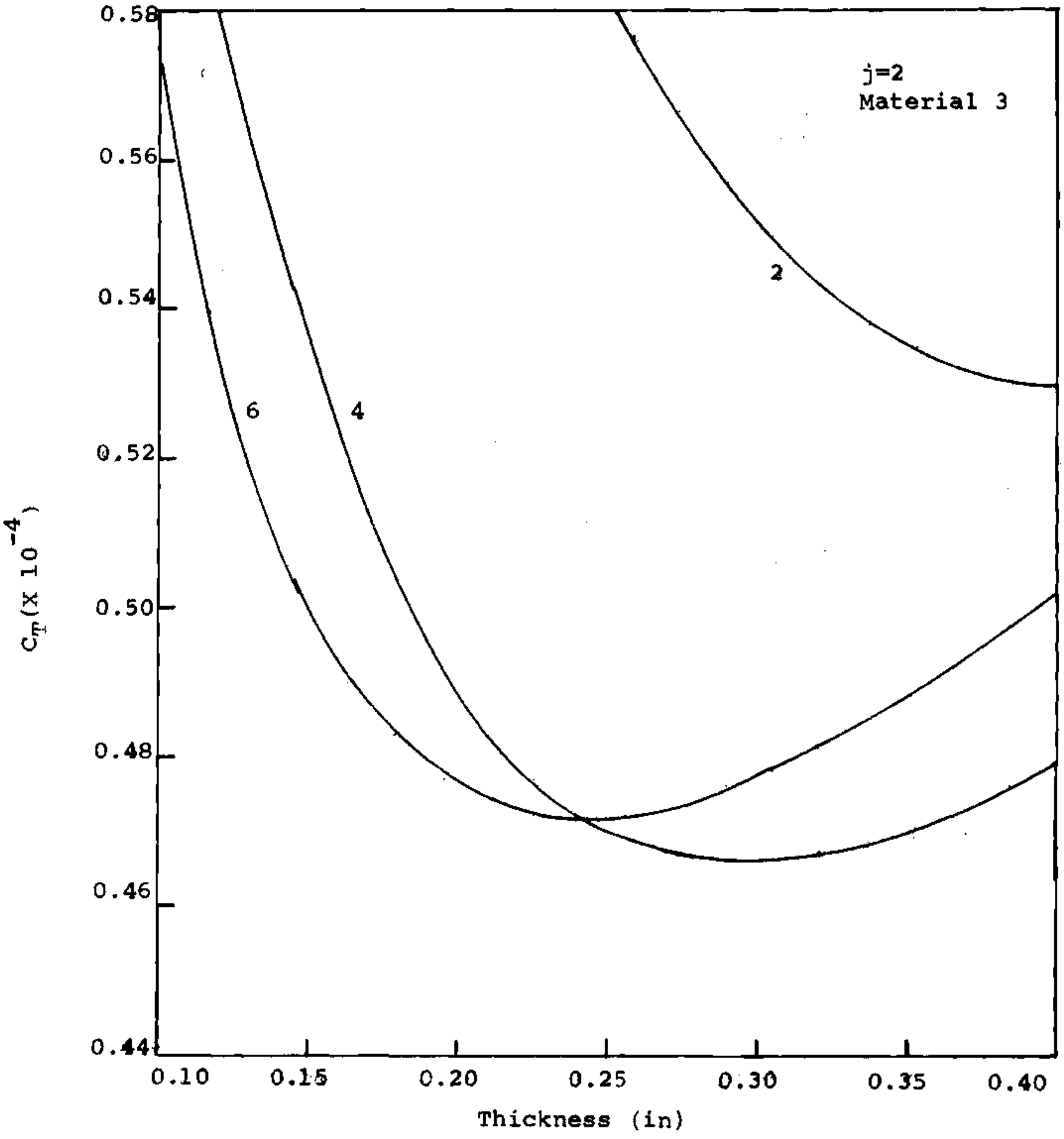

Figure 7-12. Minimization Curves

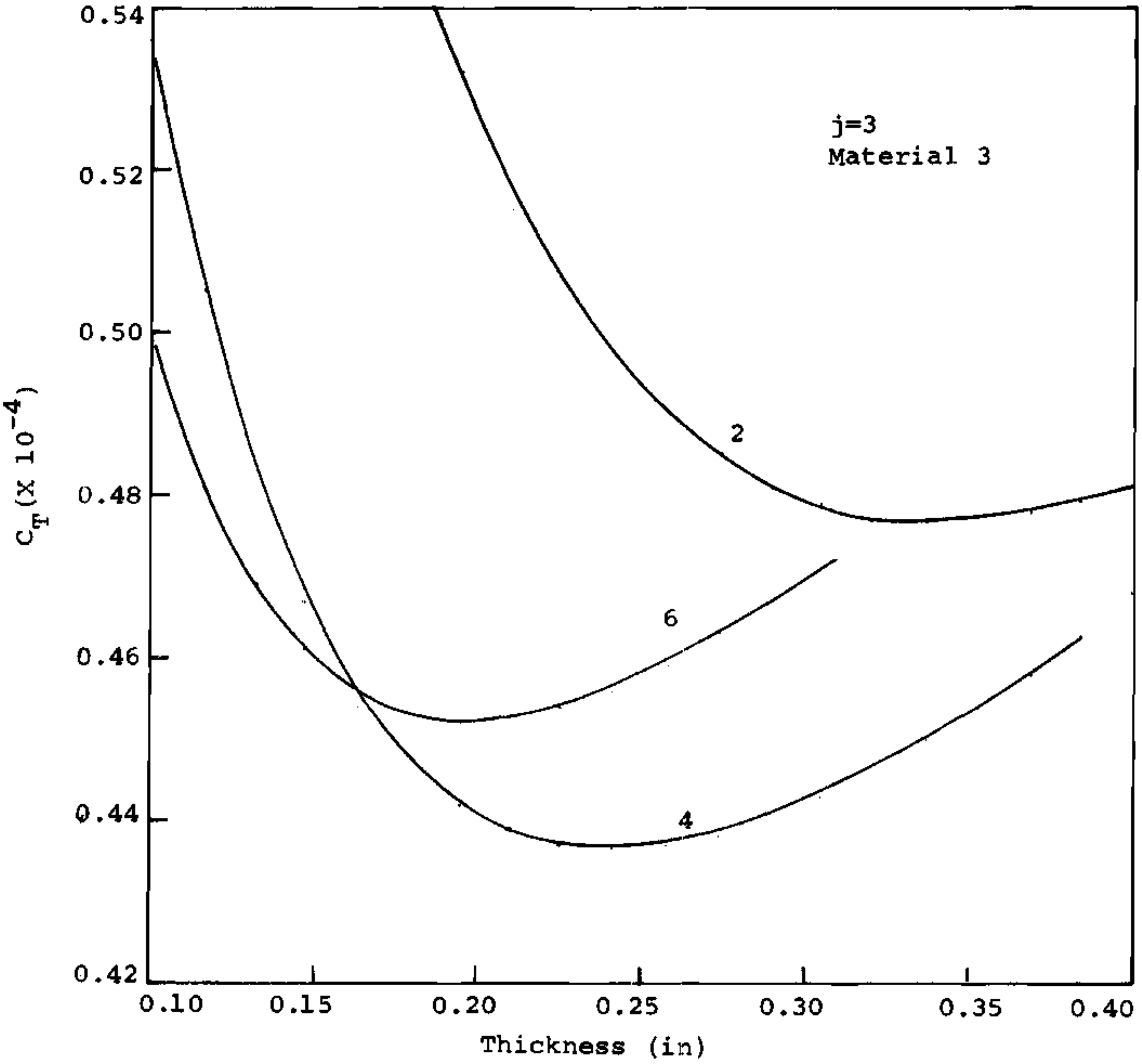

Figure 7-13. Minimization Curves

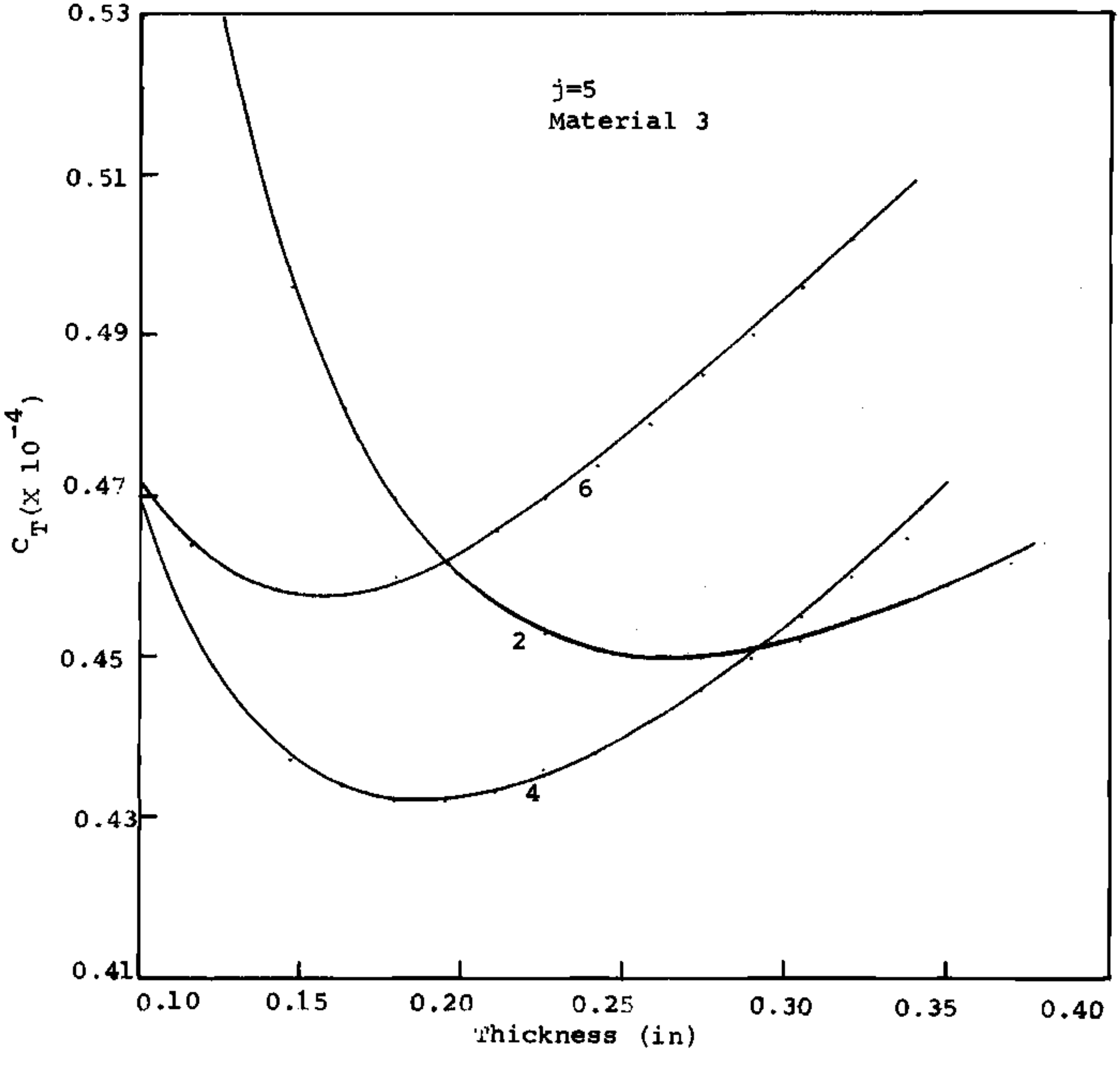

Figure 7-14. Minimization Curves

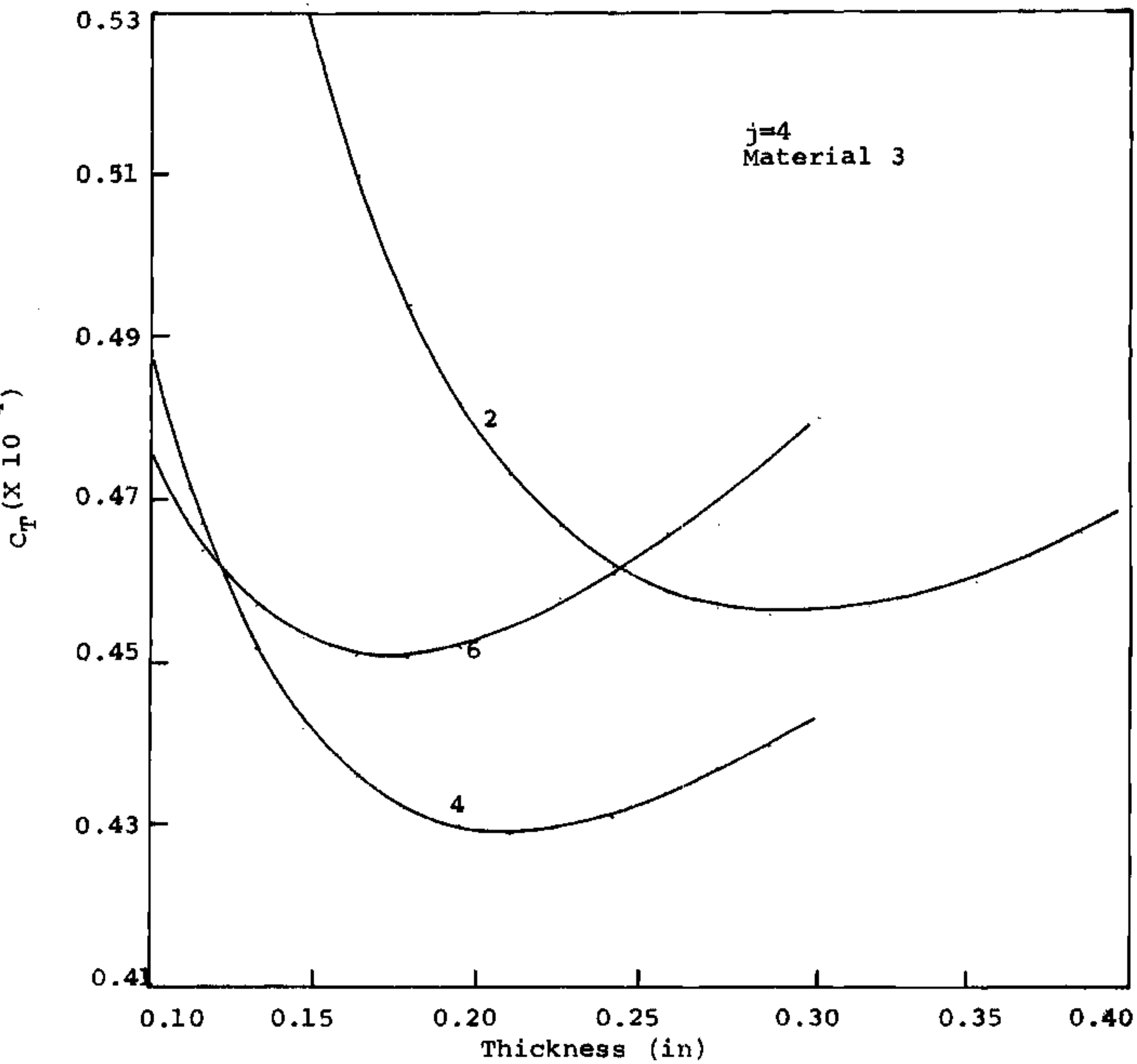

Figure 7-15. Minimization Curves

CHAPTER VIII

ILLUSTRATIVE EXAMPLE II

8-1. Problem Definition

Structural components of a solid rocket motor case are considered to be fracture critical whenever the game plan is to recover and reuse the motor case for a designated number of missions. Proof tests, conducted on the case between missions, are also significant to rendering the structural components fracture critical. Proof load levels may significantly affect the design life of the structure. A failure prevention plan is, therefore, necessary and are considered in the design of the case.

In particular, this thesis is concerned with the fracture control of the most critical membrane areas of the case. All discussions and methodologies presented in this thesis can, however, be used whenever similar fracture critical structures of a reusable vehicle system are designed. Some modification might be necessary in particular structures. Significant loads are applied to the motor case during flight and water recovery operation of each mission. The applied stresses from all other events during the mission are assumed in this analysis to be not significant enough to result in cyclic or time dependent crack growth. If the test or analysis indicate the possibility of other critical loading events they can be included in the fracture control plan by extending the reported analysis. Before each mission, the case is also subjected a proof test. The loads applied during the proof tests can result in significant amount of crack growth. Grit blasting is assumed to be used between each mission. This reduces the effective depth of cracks and the thickness of the membrane by a selected amount. While the effective depth of crack is reduced, the refurbishment grit blasting operation has the effect of increasing the applied stresses. This necessitates a larger initial thickness of the membranes than that would be required otherwise. Therefore, any design of the membrane of the case must arrive at an initial wall thickness t, the thickness Δt that will be decreased between each mission and the proof load factor p. For example, a large value of initial wall thickness results in increased reliability, but results in the need for increased propellant, increased cost of operation and reduced pay load capability. On the other hand, a small initial wall thickness increases the probability of failure and the resulting loss of the reusable space vehicle system and the pay load. Therefore, there is a need for optimizing the initial wall thickness. Similar arguments can be presented to explain the need for selecting the other design variables such as t and p by optimizing the desired objective function of cost and weight.

In general, these design variables depend on the probability distribution for the initial flaw sizes present in the membrane, applied stresses during the use of the vehicle, crack growth characteristics of the material, fracture control plans, specified reliability bounds, weight and cost considerations. The paper describes a reliability-based procedure that can be used to select the design variables of a solid rocket motor case in a reusable space vehicle system by using probabilistic fracture mechanics and cost or weight considerations.

8-2. Method of Approach

It is assumed that careful Non Destructive Inspection (NDI) techniques can detect initial cracks greater than the surface length of $2c_o$ and depth of a_o with 100% success. Sometimes, it is assumed that cracks corresponding to surface length $2c = 0.1$ inch can be identified 100% of the time [110]. If the corresponding maximum depth is 0.05 inch there is no possibility of existence any initial cracks of depth larger than 0.05 inch. Such an initial crack depth distribution is assumed to be analytically represented by Johnson S_b distribution [111]. Reasons for this assumption can be explained as follows. One of the requirements of any assumed distribution is that the minimum and maximum crack depths be bounded within finite limits. Depending on the thickness and the available techniques of non destructive inspection techniques, there is a finite maximum depth of possible crack. It is not infinity as is provided by distributions such as normal distribution, gamma or log-normal distributions. The minimum value of depth of crack can be assumed to be zero or a small number. Such a distribution can be obtained as the transformation of the usual normal variate. One such transformation is the following.

$$Z = \gamma+\eta \ln \frac{x-\varepsilon}{\lambda+\varepsilon-x} \qquad \varepsilon \leq x \leq \varepsilon+\lambda \tag{8-1}$$

In this equation, z is the standard normal variable and x is the variable of interest i.e., the crack depth. The four available parameters are γ, ε, λ, η. The minimum and maximum crack depths fix ε, λ, respectively. The parameters λ, η can be called shape parameters and can be determined from percentiles of the observed data.

The density function for the probabilistic model is written as follows

$$f_{a_o}(a_o) = \frac{\eta}{\sqrt{2\pi}} \frac{\lambda}{(a_o-\varepsilon)(\lambda-a_o+\varepsilon)} \exp$$

$$\{-\frac{1}{2}[\gamma+\eta \ln (\frac{a_o-\varepsilon}{\lambda-a_o+\varepsilon})]^{\frac{1}{2}} \tag{8-2}$$

$$\varepsilon \leq a_o \leq \varepsilon+\lambda, \qquad \eta>o$$

$$-\infty \leq \gamma \leq \infty, \qquad \lambda>o$$

$$-\infty \leq \varepsilon \leq \infty$$

This emperical distribution is called Johnson S_b distribution. It should be noted that it is possible to obtain other emperical distributions to represent the crack depths.

This probability distribution for initial crack depth changes after each mission, each proof test and each time the material as removed from the wall thickness. The change in distribution after each mission and each proof test is due to the crack growth resulting from the applied stresses. This crack growth also depends on the present length of the crack, applied stress and the material that are responsible for the crack growth. In this analysis, the applied stresses and material properties are assumed to be known deterministically. If the initial crack length were also known deterministically the crack length after each use can be determined from equations such as Paris' equation [112], Foreman's equation or Collipriests equations [113]. Because initial crack lengths are not known deterministically, crack length after each use of the vehicle is again another probabilistic distribution that has to be estimated.

The cumulative density function (CDF) for crack length after N uses is denoted by $F(a_N)$. This represents the probability that a_N A after N uses. Each use is defined as one flight, one proof test and a material removal. Crack growth due to time related effects such as stress corrosion have been neglected.

If $F(a_N)$ is known, the probability distribution for the stress intensity factor K_N can be obtained from the knowledge of the applied stresses. The probability distribution for stress intensity factor can be used to estimate the probability failure P_f which is the probability of stress intensity factor K greater than or equal to the critical stress intensity factor during the projected design life of the structure. The critical stress intensity factor is denoted by K. In this analysis, stresses and the material properties are assumed to be known deterministically. However, the applied stress changes after each use due to material removal. Therefore, the probability of failure can be expressed as the probability of a_N a_c. In this expression a_c is the critical crack depth that can be obtained from the critical stress intensity factor and the applied stress corresponding to that particular mission. This relationship between the stress intensity and the applied stress is discussed in the next section.

8-3. Stress Intensity Factor

For the analysis of the stress intensity factor in the membrane, an infinite plate model with elliptical surface flaws that are oriented perpendicular to the applied stress has been assumed. The relationship between the stress intensity factor, the applied tensile stress and crack depth is given by

$$K = \frac{1.2\ \pi\sigma^2 a}{Q(\frac{a}{c})} \tag{8-3}$$

where

$$Q(\frac{a}{c}) = \phi^2 - 0.212\ (\frac{\sigma}{\sigma_y})^2 \tag{8-4}$$

In this equation, σ_y is the yield stress and ϕ is a function of the ratio of crack depth to crack length (a/c). Variation ϕ^2 with (a/c) is given in reference [110].

Because the crack depth 'a' is a random variable the stress intensity factor K is also a random variable. In general, both crack depth 'a' and crack length '2c' are random variables and there is a need for a joint distribution for 'a' and 'c'. In this analysis, only the crack depth is considered as the random variable. It is also assumed that the probability distribution for crack depth a is known initially and is given by a Johnson S_b distribution [111]. The density function for the distribution is given in equation (8-2). This probability distribution for crack depth changes with use. The next step will be to determine the change and the new probability distribution after each flight and proof test.

8-4. Probability Distributions for Crack Depth After Use

The following symbols are used to properly account fo the changes in probability distributions.

$f(a_o)$: Probability density function for the initial crack depth

$F(a_o)$: Cumulative distribution function for initial crack depth

$F(a_{op})$: Cumulative distribution function for initial crack depth after the first proof test

$F(a_N)$: Cumulative distribution function after N flights and (N+1) tests

$F(a_{Np})$: Cumulative distribution function after N flights and N proof tests

$F(a_N)$: Cumulative distribution function after material removal from the wall thickness

The rate at which crack depth increases is assumed to be given by Paris' equation. Then,

$$\frac{da}{dN} = C(\Delta K)^n \tag{8-5}$$

where C and n are empirical constants. Alternately, the rate of crack growth can be assumed to be given by Foreman's equation or Collipriest's equation, if they are found to represent the situation more accurately. For example, Collipriest's equation can be written as follows:

$$\frac{da}{dN} = D \exp [n \frac{\ln K_c - \text{Ln } \Delta K}{2} \tanh^{-1}$$

$$\{\frac{\ln\Delta K - \frac{1}{2} (\ln K_c (1-R) + \ln\Delta K)}{\frac{1}{2} (\ln K_c \ (1-R) - \ln K)}\}$$

$$+ \ell n \ \{c \exp \ (n \frac{\ln K_c + \text{Ln } K_o}{2})\}] \tag{8-6}$$

where n is an empirical constant. By integrating either of the selected equations (8-5) or (8-6) crack depth after N+1 uses can be determined if the crack depth after N uses and N proof tests are known deterministically, i.e.,

$$a_{N+1} = a_{N+1} \{a_{Np}\} \tag{8-7}$$

Similarly, crack depth after the proof test can be determined from equation (8-5) or (8-6) if the crack depth before proof test is known deterministically, i.e.,

$$a_{Np} = a_{Np}\{a_N\} \tag{8-8}$$

These functions represented by equations (8-7) or (8-8) can be determined analytically or in the form of quadratures from equation (8-5) or (8-6). From eq. (8-7), a_{N+1} can be obtained for every known value of a_{NP}. Similarly, a_{NP} can be obtained for every known value of a_N from equation (8-8). However, both

a_{NP} and a_N are random variables in the present analysis. In this case, equation (8-7) can be used to obtain the probability distribution for a_{N+1} if the probability distribution for a_{NP} is known by using the principle of transformation of random variables. It should be noted that all equations similar to (8-7) or (8-8) involving crack depths are increasing functions. This property is useful in transforming the random variables.

For example, the probability density function for a_{N+1} can be written as follows

(8-9a)

similarly

$$f(a_{NP}) = f[a_{NP}\{a_N\}] \left|\frac{da_N}{da_{NP}}\right| \qquad (8\text{-}9b)$$

Equations (8-9a) and (8-9b) can be written for every value of N from zero to the projected number of uses.

Details of obtaining these equations for the membrane of the solid rocket motor case, with the expression for stress intensity given by equation (8-2) and Paris' equations for crack growth, is discussed in Section 8-6.

The next step is to obtain a tool for change of probability distribution due to the material removal from the wall thickness.

Material Removal and the Change of Probability Distribution

Due to material removal after each use, the effective crack depth is reduced by Δt. Thus, new crack depth is

$$\bar{a}_N = a_N - \Delta t \tag{8-10}$$

It is assumed that is a constant. Thus, by using the principles of transformation of random variables, the probability density function for a_N can be written as follows.

$$p(\bar{a}_N) = f(\bar{a}_N + \Delta t) \tag{8-11}$$

In this equation, $p(\bar{a})$ represents the density function for a_N and f represents the functional form of the probability density function for a_N.

8-5. Probability of Failure

By following the method discussed in the preceding two sections probability density function for crack depth can be obtained after every flight, proof test and material removal. From the density function, cumulative probabilities can be obtained by integration. Integration after the transformation of variables as discussed in equations (8-9), (8-10), and (8-11) needs the determination of appropriate limits of integration consistent with the transformation of variables.

If $F(a_N)$ represents the cumulative density function after flights and N proof tests the probability of failure is given by the probability of $a \geq a_{cn}$. The quantity of a_{cn} corresponds to K_c and the applied stress at the N^{th} use.

It is to be noted that the probability of failure changes with different selections of the initial wall thickness t, increased loading due to proof test, the material removed Δt and the number of designated number of missions. The increased loading due to proof tests is denoted by a factor p. A cost function or a weight function can be formulated from this knowledge of probability of failure and other related unit-cost or weight. Such a cost or weight function depends on t, p, and the number of missions N. It is possible to select these design variables by minimizing the cost or weight function subject to appropriate reliability bounds. The effect of non destructive inspection (NDI) is indirectly related to initial flaw distribution. Additional NDI effects such as the rejection of structures are not considered in the analysis. However, they can be included as cost units related to the probability of failure. A numerical example is illustrated to illustrate these developments.

8-6. Crack Growth Rate

The rate at which the crack depth increases is given by Paris' equation as follows.

$$\frac{da}{dN} = c\ (\Delta K)^n = .847x10^{-18}(\Delta K)^n$$

For subsequent convenience in algebra, the value of n is taken to be 3.0. The suggested value from current state of art is 2.48 and c is equal to $0.847x10^{-18}$. By substituting for

$$\frac{da}{dN} = 0.847\ [C_4\ \{\frac{a}{C_5+C_2\ \frac{a}{c} + C_3\ (\frac{a}{c})^2}\}^{1/2}]^3 \tag{8-12}$$

Simplifying this further,

$$\frac{da}{dN} = C_6\ \{\frac{a}{C_5+C_2\ (\frac{a}{c}) + C_3\ (\frac{a}{c})^2}\}^{1.5} \tag{8-13}$$

where

$$C_6 = 0.847\ x\ C_4{}^3\ x\ 10^{-18} \tag{8-14}$$

Separating the variables a and N in $\frac{da}{dN}$, it follows that

$$dN = \frac{1}{C_6}\ \frac{C_5+C_2\ (\frac{a}{c}) + C_3\ (\frac{a}{c})^2}{a}^{1.5}\ da \tag{8-15}$$

Integrating both sides between state (1) and state (2) the following equation is obtained

$$[N]_1^2 = \frac{1}{C_6} \int_{a_1}^{a_2} \frac{\left\{C_5 + C_2 \left(\frac{a}{c}\right) + C_3 \left(\frac{a}{c}\right)^2\right\}^{1.5}}{a} \, da \tag{8-16}$$

In order to evaluate the integral on the right hand side, it is found necessary to expand the numerator of the integrant binomially.

Now consider the numerator of the integrant with $C_5 = 1$. Neglecting terms of higher order than $(a/c)^3$, it follows that

$$\begin{aligned}\{1 + C_2 \left(\tfrac{a}{c}\right) + C_3 \left(\tfrac{a}{c}\right)^2\}^{1.5} &= 1.0 + 1.5\ C_2 \left(\tfrac{a}{c}\right) \\ &+ [1.5\ C_3 + 1.5\ (0.25)] \left(\tfrac{a}{c}\right)^2 \\ &+ [0.75\ C_2 C_3 - 0.25\ (0.5)^2\ C_2^{\ 3}] \left(\tfrac{a}{c}\right)^3\end{aligned} \tag{8-17}$$

Letting

$$P_1 = \frac{1}{c}\ 1.5\ C_2 \tag{8-18}$$

$$P_2 = \frac{1}{c^2} \{1.5\ C_3 + 0.375\ C_2^{\ 2}\} \tag{8-19}$$

and

$$P_3 = \frac{1}{C^3}\{0.75\ C_2C_3 - (0.25)^2 C_2{}^3\} \tag{8-20}$$

Then, it follows that

$$\{1+C_2\ (\tfrac{a}{c}) + C_3\ (\tfrac{a}{c})^2\} = 1.0 + P_1 a + P_2 a^2 + P_3 a^3 \tag{8-21}$$

Substituting in the integral the following result is obtained

$$[N]_{N_1}^{N_2} = \frac{1}{C_6}\ [-\frac{1}{0.5}\ (a)^{-0.5} + \frac{P_1}{0.5}\ (a)^{0.5}$$

$$+ \frac{P_2}{1.5}\ (a)^{1.5} + \frac{P_3}{2.5}\ a^{2.5}]_{a_1}^{a_2} \tag{8-22}$$

8-7. Solution of a_1 as a Function of a_2

Substituting the limits of integration in (8-22)

$$C_6(N_2-N_1) = -2a_2^{-0.5} + 2P_1(a_2)^{0.5} + \frac{2}{3}\ P_2 a_2{}^{1.5}$$

$$+ \frac{2}{5}\ P_3 a_2{}^{2.5} + 2a_1^{-0.5} - 2P_1 a_1{}^{0.5} - \frac{2}{3}\ P_2 a_2{}^{1.5} - \frac{2}{5}\ P_3 a_1{}^{2.5} \tag{8-23}$$

Rearranging and neglecting terms of order higher than three, it reduces to the following equation

$$(a_1)^3 + p(a_1)^2 + q(a_1) + r = 0 \tag{8-24}$$

where

$$p = \frac{1.0}{\frac{8}{3} P_1 P_2 - \frac{8}{5} P_3} (4P_1{}^2 - \frac{8}{3} P_2) \tag{8-25}$$

$$q = \frac{-1.0}{\frac{8}{3} P_1 P_2 - \frac{8}{5} P_3} (8P_1 + C_1{}^2) \tag{8-26}$$

and

$$r = \frac{4}{\frac{8}{3} P_1 P_2 - \frac{8}{5} P_3} \tag{8-27}$$

Now, the three roots of this cubic equation, $(a_1)^i$ are given as follows

$$a_1^{(1)} = A+B - \frac{P}{3}$$

$$a_2^{(2)} = - \frac{A+B}{2} + \frac{A-B}{2} \sqrt{-3} - \frac{P}{3} \tag{8-28}$$

$$a_1^{(3)} = - \frac{A+B}{2} - \frac{A-B}{2} \sqrt{-3} - \frac{P}{3}$$

where

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{\bar{a}^3}{27}}}$$

$$B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{\bar{a}^3}{27}}} \qquad (8\text{-}29)$$

$$\bar{a} = \frac{1}{3}(3q-p^3), \quad b = \frac{1}{27}(2p^3-9pq+27r)$$

8-8. Transformation

Probability density of a_2 is given by

$$f_{a_2}(a_2) = \frac{da_1}{da_2} f_{a_1}(a_1) \qquad (8\text{-}30)$$

CDF of a_2 is then

$$\int_{\bar{a}_1}^{\bar{a}_2} f_{a_2}(a_2)da_2 = \int_0^{a_1(a_2)} f_{a_1}(a_1)da_1$$

$$\int_0^{a_1(a_2)} f_{a_1}(a_1)da_1 = [F_{a_1}^{(a_1)}]_0^{a_1(a_2)} \qquad (8\text{-}31)$$

where $F_{a_1}(a_1)$ is the CDF of Johnson S_B distribution.

Now, it is needed to obtain a_1 as a function of a_2, No. of cycles, etc. This can be done by solving the polynomial equation obtained previously in terms of a_1 and treating a_2, N_1 and N_2 as constants. The infinite degree polynomial equation is truncated at the 3rd degree for convenience.

Of the three roots only one will be the real root because of the physical nature of the problem, say $a_1(a_2)$.

Then by substituting in the expression for the CDF of a_2

$$F_{a_2}(a_2) = \int_{o}^{a_1(a_2)} f_{a_1}(a_1)da_1 \tag{8-32}$$

or if the CDF of a_1 is known,

$$F_{a_2}(a_2) = [F_a(a_1)]_o^{a_1(a_2)} \tag{8-33}$$

Thus, $F_{a_2}(a_2)$ is a function of the parameters of flaw distribution, i.e., ε, λ, ν, η, the proof test factor p and the number of uses (N_2-N_1).

The effect of each of these parameters can be studied by calculating $F_{a_2}(a_2)$ for various cases, by means of a computer.

8-9. Parabolic Fit to $\phi^2 (\frac{a}{c})$

Consider the range $0 \leq \phi^2 \leq 1$. In this range it is attempted to fit a parabolic curve for such as follows.

$$\phi^2 (\tfrac{a}{c}) = \bar{C}_1 + \bar{C}_2 (\tfrac{a}{c}) + \bar{C}_3 (\tfrac{a}{c})^2 \tag{8-34}$$

In order to determine the three constants C_1, C_2 and C_3 three points are considered on the given curve.

$$\frac{a}{c} = 0 \quad \phi^2(\frac{a}{c}) = 1.0 \tag{8-35}$$

$$\frac{a}{c} = 0.5 \ \phi^2(\frac{a}{c}) = 1.5$$

$$\frac{a}{c} = 1.0 \ \phi^2(\frac{a}{c}) = 2.5$$

Substituting the values for point (ii)

$$1.0 + \tilde{C}_2 \ (0.5) + \tilde{C}_3 \ (0.25) = 1.5 \tag{8-36}$$

or

$$2\tilde{C}_2 + \tilde{C}_3 = 2.0 \tag{8-37}$$

Substituting the values for point (iii)

$$1.0 + \tilde{C}_2 + \tilde{C}_3 = 2.5 \tag{8-38}$$

or

$$\tilde{C}_2 + \tilde{C}_3 = 1.5 \tag{8-39}$$

Solving equations (8-2) and (8-3) simultaneously

$$C_2 = 0.5 \tag{8-40}$$

and

$$C_3 = 1.0 \tag{8-41}$$

Thus the chosen parabolic fit is as follows

$$\phi^2 = 1.0 + 0.5 \frac{a}{c} + \frac{a^2}{c^2} \tag{8-42}$$

8-10. Limits of Integration for the CDF of a_2

By hypothesis, the initial flaw a_1 has a Johnson S_b distribution. Also there is a functional relationship between the initial flaw size a_1 and the subsequent flaw size a_2 after N cycles. This relationship renders a_2 a random variable because a_1 is a random variable by hypothesis. Having known the range space of a_1 the range space of a_2 can be derived from the functional relationship between a_1 and a_2. Thus, if the lower limit of a_1 is zero, it follows from the functional relationship between a_1 and a_2 that the lower limit of a_2 is also zero. Next, if the upper limit of a_1 is a_1, the corresponding upper limit for a_2 can be obtained by solving the cubic relation between a_1 and a_2, as a function of the number of cycles $N^2 = -N_2 - N_1$.

8-11. Numerical Example

For the numerical example, it is assumed that the Johnson S_b distribution for the initial crack depth is such that the minimum crack depth is zero and the maximum crack depth is 0.1 inch. Paris's equation for crack growth is assumed with

$$c = 0.847 \times 10^{-18}$$
$$n = 3.0 \tag{8-43}$$

The variation of ϕ^2 with (a/c) as shown in Figure 8-1 is approximated by a quadratic relation.

The primary objective of reusing the solid rocket motor case is to reduce the cost of operation of the reusable space vehicle system in which it is used. However, as the number of uses is increased, the probability of failure increases because of the propagation of the crack depth. On the other hand, smaller number of uses increases reliability and also the cost is distributed over a smaller number of uses. This means the casing has to be replaced after a fewer number of uses.

A larger initial thickness would increase the weight of the casing and costs more in terms of payload. But the probability failure is less if the thickness is more. The proof test factor and the material erosion are kept constant in this example. However, they also can be varied and their

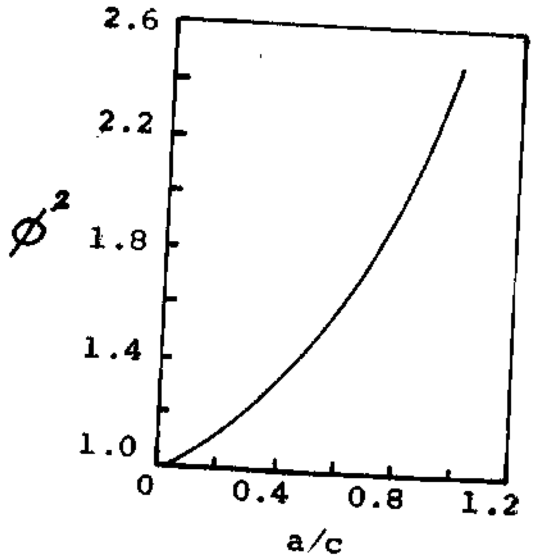

Figure 8-1. Shape Factor

effect on total cost can be considered in the most general case. The total cost function c_T, therefore, comprises the following component costs.

(i) Initial cost of the casing, c_i,

(ii) Expected cost of flight failure c_{ii},

(iii) Expected cost of proof test failure c_{iii}, and

(iv) Cost due to multiple usage, c_{iv}.

The initial cost c_i is given by the product of the weight of the casing and the cost per pound of the system, i.e.,

$$C_i = \pi(2R_o t_N - t_N^2) H \gamma C_1 \tag{8-44}$$

where

R_o = outer radius of the casing

t_N = thickness of the casing at the N^{th} cycle

H = height of the casing

γ = density of the material

C_1 = payload cost per pound

The expected cost of flight failure is the product of the probability of flight failure and the entire payload cost, i.e.

$$C_{ii} = P_N \cdot C_2 \tag{8-45}$$

where P_N is the probability of failure at the N^{th} flight and c_2 is the total cost of the payload. Similarly the cost of proof test failure is

$$C_{iii} = p_{np}\, C_3 \tag{8-46}$$

where p_{np} is the probability of failure at the N^{th} proof test and c_3 is cost of articles and accessories of proof test. Finally, the cost due to multiple usage is given as follows:

$$C_{iv} = C_3/(N)^{0.3} \tag{8-47}$$

Thus, substituting all the components, the total cost function c_T is given by the following equation

$$c_T = C_i + C_{ii} + C_{iii} + C_{iv} \tag{8-48}$$

The following numerical values are used[1,6] in evaluating equation (8-48).

γ = 0.3 lbs/cubic inch

H = 816 inches

R_o = 72.5 inches

C_1 = \$1624 per lbs

C_2 = $\$250 \times 10^6$

C_3 = $\$2 \times 10^6$

8-12. Results

The initial thickness to is varied from 0.535" to 0.435" in steps of 0.005". Also 1% of the initial thickness is eroded after each flight. The total cost function is calculated for various initial thicknesses and use cycles by means of a digital computer. Figure 8-2 illustrates the variation of the cost function with t_o and N. If it is obvious that as the number of uses increases, the minimum occurs at a higher initial thickness. For example, for 18 missions the minimum cost occurs at an initial thickness of 0.48. The initial thickness to give minimum cost for 20 mission cycles increases to 0.497, for 22 missions the thickness required is 0.512".

Figure 8-3 delineates the variation of reliability with initial thickness, after 20 missions cycles. The reliability corresponding to the minimum cost for 20 uses is 99.3%. If this reliability is not adequate enough, then a higher initial thickness should be used even though the total cost will be higher than the minimum.

General Procedure

Based on the preceding example, a general procedure can be delineated in the following steps.

1. Obtain the parameters of the Johnson S_b distribution for the initial flaw size.

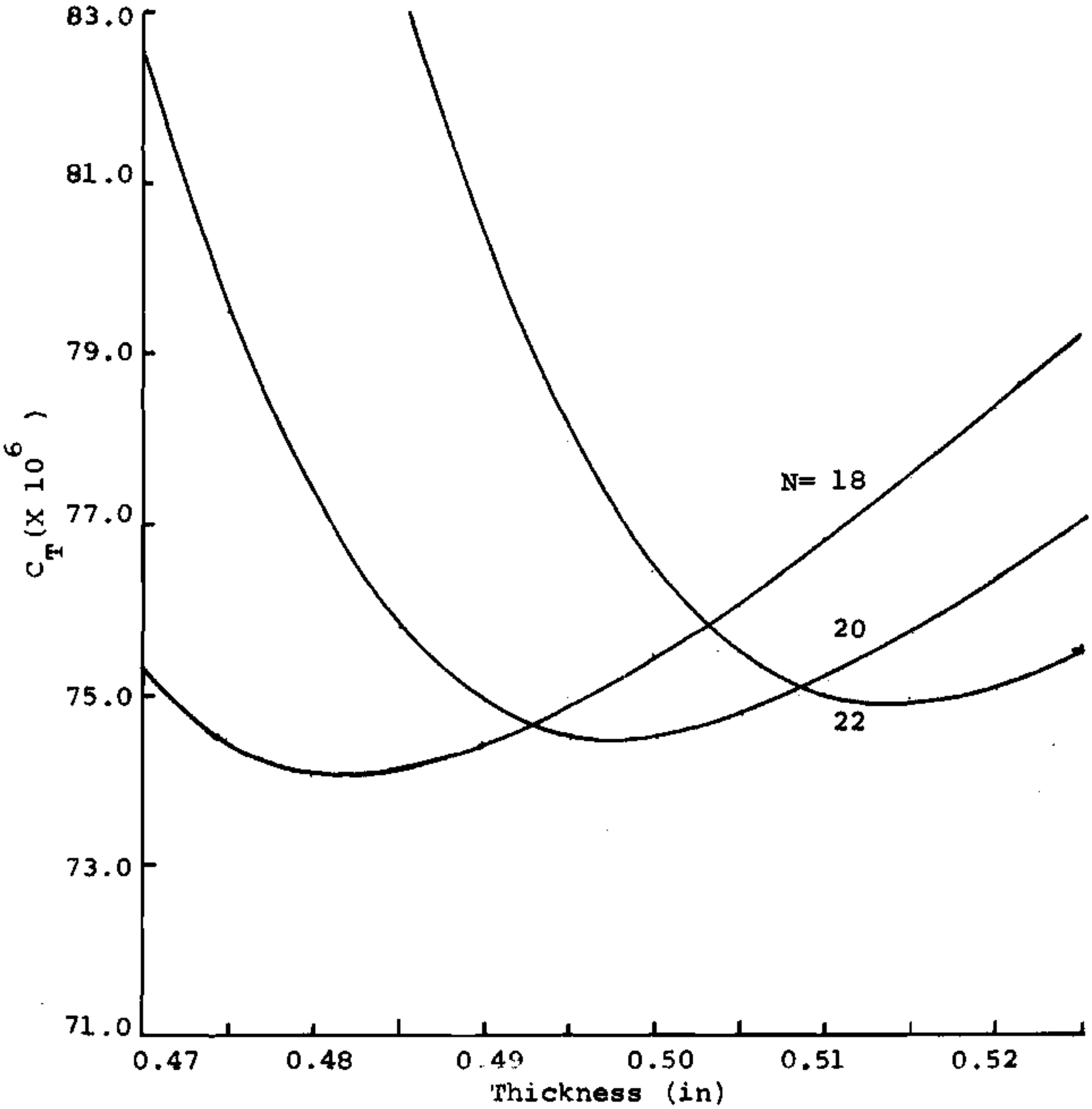

Figure 8-2. Minimization Curves

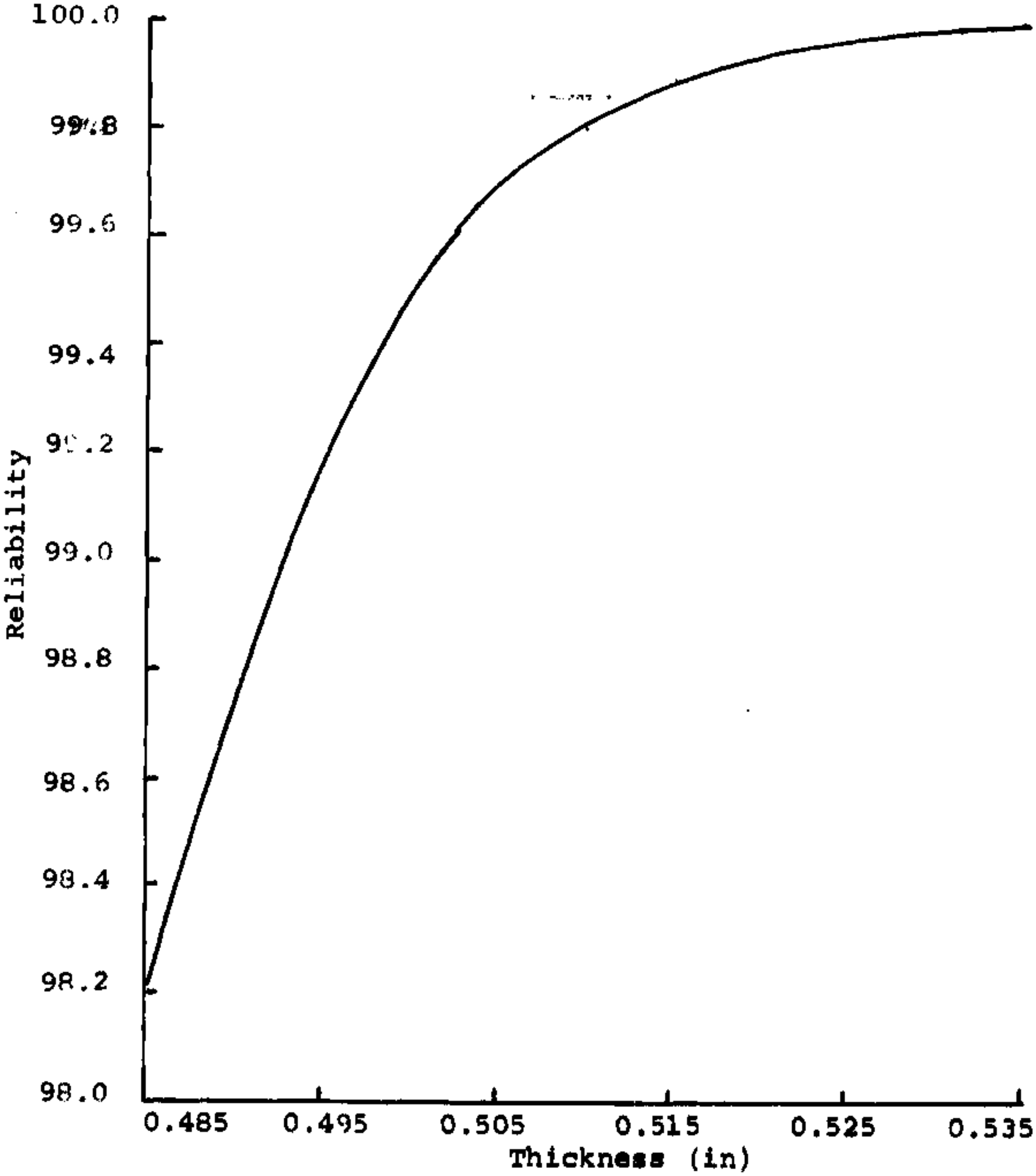

Figure 8-3. Reliability After 20 Launch Cycles

2. Obtain the stress in the membrane from the known geometry of the case and wall thickness.

   In the equation p is the proof stress factor. During flight, p is replaced by a value of 1. Pressure P is the pressure and R_o is the radius of the case.
3. Obtain the new CDF and density function for the crack depth after the proof test.
4. Obtain the new CDF for the crack depth during the flight following the proof test.
5. Estimate the probability of failure.
6. Compute the cost function parameters.
7. Obtain the new CDF after the material removal.
8. Repeat steps 2 to 7 for the new thickness and the next mission until the total number of missions are complete.
9. Change t and N and repeat the calculations as necessary.
10. Select the design variables for the minimum value of the objective function subject to reliability constraints.

A computer program has been written to carry out these steps.

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

The field of research discussed in this thesis encompasses theories of probability, random loading, fracture mechanics and fatigue design. These diverse theories are unified in a consistent manner as demonstrated by the two illustrative examples. The first example was "optimum reliability based fail-safe fatigue design of a simple aircraft structure." The second illustrative example was "optimum fracture control procedure for a reusable solid rocket motor case." In brief the following is the glossary of conclusions from the entire research project.

The preliminary analysis of the fatigue data points out that a stochastic process fatigue is the only way to realistically model fatigue in aircraft structures. The stochastic model developed during the research encompasses the fatigue damage from crack initiation to fracture. It possesses the freedom of hypothesizing a model for initiation by varying the discrete length unit $\Delta\ell$. The model is useful in optimizing the inspection, repair and maintenance schedule for fatigue damage of aircraft structures. The thesis demonstrates how the developed reliability theory can be useful in practical designs in selecting optimum (i) geometry,

(ii) material selection, and (iii) maintenance schedule.

The numerical multiple integration technique with respect to one independent variable that is developed here is more accurate than the original [98]. The three numerical examples investigated are

(i) Integration of an algebraic function

(ii) Integration of homogeneous differential equation (free vibration of a beam) and

(iii) Integration of non-homogeneous differential equation (forced vibration problem).

All the three examples indicated better accuracy with the proposed technique, (Chapter V).

Scope for Further Research

(1) The discretization in crack length that is introduced in the stochastic model may be removed so that the crack length is a continuous random variable.

(2) Development of a similar stochastic model for multiple cracks is another prospect.

(3) Quantitative estimation of the model and test of significance.

(4) Design of experiments to estimate the model.

(5) Improved cumulative damage theory based on the model.

(6) Development of a joint probability distribution for range (ΔL) and ratio (r) of minimum load and maximum

load in random fatigue loading characterization.

(7) Minimization of the probability of failure rather than some objective function such as expected cost function or weight function.

(8) Aerospace vehicles are sometimes exposed to random temperatures. Consideration of these random thermal effects, which is definitely another source of uncertainty, in the fail-safe design procedures is a new field in itself.

(9) Application of the reliability based, fail-safe fatigue design methodology to composite structures of current and future interest in aerospace engineering.

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APPENDIX I

FATIGUE CRACK DATA--PRELIMINARY ANALYSIS

Statistical Models

Statistical modeling has been traditionally instrumental in the analysis of large accumulation of data pertaining to many physical phenomena. Bacteriology, earthquake engineering, telephone technology, electronics and mechanical engineering are some examples that can be cited. This is extended to the analysis of fatigue crack data analysis of aircraft structures. Many attempts have been made in the past to develop statistical models to describe fatigue failure qualitatively and quantitatively. Most of the investigations have been restricted to the use of coupon tests. These investigations assume certain results derived from the coupon test data, e.g., the value of the shape parameter in the Weibull distribution. Also appropriate significance tests have not been applied to ascertain that the hypothesized model accurately represents the data. In the present research fatigue crack data of the center box wing of a given fleet of aircraft is analyzed statistically. Previous researchers have agreed that Weibull distribution is most suitable to represent the fatigue life treated as a random variable. However, some modifications to the data may have to be made such as censoring, excluding runout data etc.

Fatigue Crack Data Description

Fatigue crack data of the center box wings of a fleet of aircraft is collected during service inspections. The wing has some locations which are more susceptible to fatigue damage than the rest of the wing, due to the presence of rivets, stress raisers, weldments, etc. These fatigue prone regions are numbered from 1 to 94 and are hereafter referred to as stations XX. Figures 1 and 2 illustrate the wing surfaces with the stations 1-94. Each of the wings of the fleet of aircraft is examined for fatigue cracks at these stations during periodic service inspections. If there is a crack present its particulars are recorded. Some of the relevant information in the inspection record are the following.

(1) Number of flight hours completed before the inspection

(2) Station number where crack exists

(3) Length of the crack

(4) Names of command and base

The huge volume of data collected on each of the 94 stations of the 130 aircraft is rearranged for systematic analysis.

Data Reduction

The data on each station is noted separately and is further separated by the bases where the aircraft are in operation. The base dictates the kind of activity the aircraft is subjected to and thus the nature of loading acting on the aircraft. A typical listing of the data is shown in Table 1. There is a great deal of scatter in the data. In order to examine the homogeneity of the fatigue data analysis of variance is sought. The purpose of analysis of variance is to see the possibility of lumping all the data at various stations together and fit one probabilistic fatigue model for the entire wing/aircraft. If this is feasible, the fatigue analysis of the aircraft is greatly simplified. The analysis of variance as applied to the present fatigue data and its results are presented in the following section.

Analysis of Variance

The name analysis of variance stems from an analysis in which the total variation in the data is partitioned into component parts. These components are then used to develop a test statistic. The test statistic is then compared with a theoretical critical value corresponding to a chosen significance level. If the test statistic is less than the critical value then there is no significant variation in the data at that level of significance. Then, the data is considered homogeneous otherwise it is heterogeneous. If the data is heterogenous then it has to be divided into groups of homogeneous data. This is the subject matter of the next section.

The total variation in the data is expressed by the total corrected sum of squares, i.e.

Table 1. Typical Data Record for Station B-79

Serial No.	Time (1000 hr)	Crack Length (in)	Serial No.	Time (1000 hr)	Crack Length (in)
1	4.246	0.12	29	5.620	0.12
2	4.904	0.31	30	5.374	0.12
3	4.972	0.62	31	4.292	0.06
4	4.349	0.06	32	4.091	0.06
5	5.072	0.12	33	7.332	0.06
6	4.888	0.03	34	4.505	0.12
7	4.653	0.25	35	4.574	0.12
8	5.014	0.09	36	4.623	0.12
9	4.484	0.12	37	4.100	0.06
10	4.574	0.12	38	4.893	0.09
11	4.722	0.03	39	4.148	0.19
12	4.268	0.03	40	4.719	0.09
13	4.914	0.12	41	4.432	0.03
14	4.350	0.12	42	4.358	0.03
15	3.774	0.09	43	4.710	0.03
16	4.705	0.03	44	4.630	0.06
17	4.405	0.03	45	4.691	0.25
18	4.944	0.03	46	4.239	0.12
19	3.663	0.06	47	4.215	0.25
20	4.415	0.06	48	4.516	0.06
21	4.595	0.06	49	4.478	0.12
22	4.064	0.06	50	5.110	0.03
23	3.951	0.12	51	3.757	0.06
24	5.152	0.12	52	4.516	0.03
25	3.506	0.06	53	4.396	0.06
26	4.400	0.12	54	4.682	0.06
27	4.624	0.06	55	4.627	0.15
28	4.731	0.03	56	3.877	0.06
			57	3.858	0.06

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{..})^2 \qquad (1)$$

Equation (1) can be expanded and simplified to result the following

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n_i} X_{ij}^2 - \frac{T_{..}^2}{N} \qquad (2)$$

where

$$T_{..} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} X_{ij}$$

$$N = \sum_{i=1}^{a} n_i$$

n_i = number of items in 'i'th group

and a = number of groups being lumped together.

The total variation can be split up into two components as follows

$$SS_T = \sum_{i=1}^{a} n_i (\bar{X}_{i.} - \bar{X}_{..})^2 + \sum_{i=1}^{a} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})^2 \qquad (3)$$

$$= SS_A = SS_E$$

where

SS_A = variation between groups

SS_E = variation within groups

The expression for SS_A can be simplified as follows.

$$SS_A = \sum_{i=1}^{a} n_i(\bar{X}_{i.} - \bar{X}_{..})^2$$

$$= \sum_{i=1}^{a} \frac{T_{i.}^2}{n_i} - \frac{T_{..}^2}{N} \qquad (4)$$

Equations (1) and (4) are employed in numerical calculations. A computer program has been written to perform all the routine calculations and print the results in a tabular form. The procedure for the analysis of variance is indicated in Table 2.

Table 2. Analysis of Variance

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Sum of Squares	F
Between groups	SS_A	a-1	$SS_A/(a-1)$	$\frac{SS_A/(a-1)}{SS_E/(N-a)}$
Within groups	SS_E	N-a	$SS_E/(N-a)$	
Total	SS_T	N-1		

The result of the analysis of variance of the entire data is shown in Table 3. The conclusion is that there is a

Table 3. Analysis of Variance for B(1-92) Stations

Source of Variation	Sum of Sqrs.	D.O.F.	Mean S. Sqrs.	F
Between Stations	121.0386	82	1.4761	1.9292
Within Stations	816.3774	1067	.7651	
Total	937.4160	1149		

$F_{82,\infty}$ at 10% = 1.24

5% = 1.31 < 1.9292

1% = 1.44

Data varies significantly.

significant variation in the data. Next, it is attempted to separate the data into smaller groups without significant variation within them.

Duncan's Multiple Range Test

The theoretical development of this test is not concerned here since the primary interest is its application. Basically this test consists of comparing the modified difference between the various 'means' $(m_i-m_j)'$ with the corresponding critical value R_p'. The modified means are calculated from the following expressions

$$(m_i-m_j)' = (m_i-m_j)a_{ij} \tag{5}$$

and

$$a_{ij} = [2r_ir_j/(r_i+r_j)]^{1/2} \tag{6}$$

In equation (6) r_i and r_j are the number of replications in each group in calculating the mean values m_i and m_j for the ith and jth group, respectively. The critical values R_p' can be calculated from the following equation.

$$R_p' = Z_p \cdot s \tag{7}$$

where Z_p is the studentized range for 'p' means, [Table II and III, Duncans' papers, Ref. 99,100] and S^2 is the mean sum of squares within the groups.

Multiple Range Test for Heteroscedastic Means of Groups (71-80)

Means are calculated for the groups (71-80) and ranked in the increasing order as follows.

Table 4. Ranked Means

Group No.	Mean	No. of Replications
77	3.305	(1)
75	4.2775	(2)
74	4.48021	(58)
72	4.53449	(57)
73	4.55738	(77)
71	4.6263	(66)
79	4.744	(4)
80	5.64733	(3)
78	5.8165	(2)
76	7.332	(1)

The numbers in parentheses indicate the members of replications (r) which all unequal. Then all the possible combinations of groups are subjected to Duncan's test and those groups whose modified means do not exceed the critical value of R_p' belong to one homogeneous set of data. The procedure is best explained by means of an example given below.

Application

The procedure that has been briefly presented in the preceding paragraph is now applied to stations (71-80) consisting of 10 groups, to see if they can be lumped together. To start with the test first analysis of variance table is formulated as shown in Table 4. The data shows significant variation. In order to find out which of the groups are causing the non-homogeneity the modified differences in the means of various grups are calculated. Table 5 presents the ranked mean values and the number of replications for each group. The modified differences for all the possible combinations of two groups are calculated and listed in Table 6. Next, the critical values R_p' are calculated for $p = 2, 3, \ldots 10$ and tabulated in Table 7.

Now compare the modified differences between any two means with the corresponding R_p' to see if it is not exceeding R_p'. For example,

$(76-77)' > R_{10}'$

$(76-75)' > R_9'$

$(76-74)' > R_8'$

$(76-72)' > R_7'$

$(76-73)' > R_6'$

$(76-71)' > R_5'$

$(76-79)' > R_4'$

$(76-80)' > R_3'$

$(76-78)' > R_2'$

Table 5. Analysis of Variance for B(71-80) Groups

Source of Variation	Sum of Sqrs.	D.O.F.	Mean S.Sqrs.	F
Between Stations	16.8484	9	1.8720	2.7722
Within Stations	176.2540	261	.6753	
Total	193.1025	270		

At 1% $F_{9,261} = 2.5$

5% $= 1.9$

10% $= 1.65$

Conclusion: Significant variation in data.

Table 6. Modified Differences

$$a_{i,j} = \sqrt{\frac{2r_i r_j}{(r_i + r_j)}}$$

Group	Modified Difference
(76-77)'	= 4.027 x (1)
(76-75)'	= 3.0545 x 1.1547 = 3.527
(76-74)'	= 3.999 x ( )
(76-72)'	= 3.922 x ( )
(76-73)'	= 3.896 x ( )
(76-71)'	= 3.6599 x ( )
(76-79)'	= 2.812 x ( )
(76-80)'	= 2.0637 x ( )
(76-78)'	= 1.201 x ( )
(80-77)'	= 2.3423 x ( ) = 2.8687
(80-74)'	= 1.167 x ( ) = 2.7873
(80-72)'	= 1.113 x ( ) = 2.6579
(80-73)'	= 1.0903 x ( ) = 2.6201
(80-71)'	= 1.0207 x ( ) = 2.4452
(80-79)'	= 0.903 x ( ) = 2.365
(79-77)'	= 1.439 x ( ) = 1.802
(79-75)'	= 0.4665 x ( ) = 0.7618
(79-74)'	= 0.2638 x ( ) = 0.720
(79-72)'	= 0.2096 x ( ) = 0.5731
(79-73)'	= 0.187 x ( ) = 0.5157
(79-71)'	= 0.1177 x ( ) = 0.3232

Table 6 (continued)

Group	Modified Difference
(71-77)'	= 1.3213 x ( ) = 1.8546
(71-75)	= 0.3488 ( ) = 0.6872
(71-74)'	= 0.14609 ( ) = 1.148
(71-72)'	= 0.0918 ( ) = 0.7179
(71-73)'	= 0.069 ( ) = 0.5817
(73-77)'	= 1.2525 ( ) = 1.7596
(73-75)'	= 0.2797 ( ) = 0.5522
(73-74)'	= 0.0772 ( ) = 0.6279
(73-72)'	= 0.0228 ( ) = 0.1845
(72-77)'	= 1.2295 ( ) = 1.7837
(72-75)'	= 0.257 ( ) = 0.5052
(72-74)'	= 0.0543 ( ) = 0.4117
(74-77)'	= 1.1752 ( ) = 1.6478
(74-75)'	= 0.2027 ( ) = 0.3985
(75-77)'	= 0.9725 ( ) = 1.1229

Table 7. Critical Values R_p'

p	2	3	4	5	6	7	8	9	10
Z_p	2.77	2.92	3.02	3.09	3.15	3.19	3.23	3.26	3.29
R_p	2.275	2.4	2.480	2.535	2.588	2.62	2.65	2.675	2.71

$$S = \sqrt{0.6753}$$

$$= 0.822$$

$$R_p' = Z_p \cdot s$$

Z_p values from Table IV of Duncan's paper.

This indicates that groups (76,78,80) form one homogeneous set. Proceeding in this manner with all the possible combinations, it follows that groups (76,78,80) and (71,72, 73,74,75,77,79) are two homogeneous sets of data.

The particular aircraft under consideration has 94 stations. The analysis of variance and Duncan's test results are presented in Tables 8-14.

Next it is decided to fit two parameter Weibull distribution to station or station groups lumped together if they are homogeneous.

Weibull Model

The time 'T' to fatigue failure is 'defined' as the number of flight hours completed by the inspection time if a crack length $\geq$ 0.03 inch is detected. This time 'T' to failure is a random variable because of the observed scatter in the data. It is intended to fit two-
model to the time to failure. The density function $f_T(t)$ of the hypothesized Weibull distribution is given by the following

$$f_T(t) = \frac{\alpha}{\beta} \left(\frac{t}{\beta}\right)^{\alpha-1} \exp[-(t/\beta)^{\alpha}] \tag{8}$$

where

T is the random variable for failure time

t is the particular value taken by T

Table 8. Analysis of Variance for B(1-15)

Source of Variation	Sum of Sqrs.	D.O.F.	Mean S.Sqrs.	F
Between stations	10.1206	13	.7785	.8204
Within stations	174.6106	184	.9490	
Total	184.7312	197		

At 1%--$F_{13,184} = 2.04$

5%--$F_{13,184} = 1.67$ > 0.8204

10%--$F_{13,184} = 1.55$

Table 9. Analysis of Variance for B-33-38 Stations

Source of Variation	Sum of Sqrs.	D.O.F.	Mean S.Sqrs.	F
Between stations	2.9861	5	.5972	2.2285
Within stations	9.6477	36	.2680	
Total	12.6338	41		

$$F_{5,36} \text{ at } 10\% = 2.00 \quad < 2.2285$$

$$\left.\begin{array}{r} 5\% = 2.45 \\ 1\% = 3.51 \end{array}\right\} > 2.2285$$

Table 10. Analysis of Variance for B-41-46 Stations

Source of Variation	Sum of Sqrs.	D.O.F.	Mean S.Sqrs.	F
Between stations	2.2298	5	.4460	.8528
Within stations	23.5306	45	.5229	
Total	25.7604	50		

Table 11. Analysis of Variance for B-41-46 and B9-92 Stations

Source of Variation	Sum of Sqrs.	D.O.F.	Mean S.Sqrs.	F
Between stations	3.5008	9	.3890	.4847
Within stations	270.4653	337	.8026	
Total	273.9661	346		

Conclusion: No significant variation in data.

Table 12. Analysis of Variance for B-61-72 Stations

Source of Variation	Sum of Sqrs.	D.O.F.	Mean S.Sqrs.	F
Between stations	33.6304	9	3.7367	5.4579
Within stations	20.5392	20	.6846	
Total	54.1696	39		

$$F_{9,30} \text{ at } \left.\begin{array}{l} 10\% = 1.85 \\ 5\% = 2.27 \end{array}\right\} < 5.4579$$
$$1\% = 3.07$$

Conclusion: Data varies significantly.

Table 13. Analysis of Variance for B(33-38,41-46,61-72, 89-92)

Source of Variation	Sum of Sqrs.	D.O.F.	Mean S.Sqrs.	F
Between stations	95.7112	25	3.8284	5.1131
Within stations	301.7499	403	.7488	
Total	397.4611	428		

$F_{25,\infty}$ at 1% = 1.79

5% = 1.52 < 5.1131

10% = 1.33

Table 14. Analysis of Variance for B-89-92 Stations

Source of Variation	Sum of Sqrs.	D.O.F.	Mean S.Sqrs.	F
Between stations	1.2671	3	.4224	.4994
Within stations	246.9359	292	.8457	
Total	248.2030	295		

Conclusion: No significant variation in data.

α is the shape parameter, and

β is the scale parameter.

The values of the two parameters α and β can be estimated from the available data by the classical maximum likelihood estimation method.

Maximum Likelihood Estimation

The maximum likelihood method tentatively assumes that (1) the parameters of the hypothesized distribution are known, and (2) the observed values are independent random variables having the same parameters and distributions. Then, the so called likelihood function is defined as the joint density function of all these random variables. The parameters are then estimated by maximizing the likelihood function with respect to the unknown parameters, which were assumed known to write the joint probability density function.

Let T_i, i = 1, 2, ..., n, n+1, ..., k be the random variables corresponding to the observed failure times t_i, i = 1, 2, ..., n, and run-out data (cracks are not existing at the inspection time) t_j, j = n+1, n+2, ..., k. The probability density functions of these independent random variables are as given below.

$$f_{T_i}(t_i) = \frac{\alpha}{\beta} \left(\frac{t_i}{\beta}\right)^{\alpha-1} \exp[-(t_i/\beta)^{\alpha}], \quad i=1,2,\ldots,k \tag{9}$$

and the cumulative distribution function is

$$F_{T_i}(t_i) = 1-\exp[-(t_i/\beta)^{\alpha}] \tag{10}$$

Since all the T_i are independent random variables, their joint probability density function is given by the product of all the individual density functions. This quantity is defined as the likelihood function L, i.e.

$$L(\alpha,\beta|T_i=t_i,\ T_j>t_j)$$

$$= \prod_{i=1}^{n} f_{Ti}(t_i) \prod_{j=n+1}^{k} (1-F_{T_j}(t_j), \quad i=1,2,\ldots,n \quad j=n+1,n+2,\ldots,k \tag{11}$$

Substituting from equations (3-9) and (3-10) into equation (11)

$$L = \prod_{i=1}^{n} \frac{\alpha}{\beta} \left(\frac{t_i}{\beta}\right)^{\alpha}\exp[-(t_i/\beta)^{\alpha}] \prod_{j=n+1}^{k} \exp[-(t_j/\beta)^{\alpha}] \tag{12}$$

By taking logarithm of equation (12) all the product symbols π become summation symbols Σ which is easier for subsequent differentiation partially with respect to α and β. Mathematically the maximization is not affected because the function L and its logarithm have the maximum at the same point. Then,

$$\ell n(L) = \sum_{i=1}^{n} [\ell n \ (\frac{\alpha}{\beta}) + (\alpha-1) \ \ell n(\frac{t_i}{\beta}) - (\frac{t_i}{\beta})^{\alpha}]$$

$$- \sum_{j=n+1}^{k} (t_j/\beta)^{\alpha} \tag{13}$$

The necessary conditions for ℓn L to be maximum are

$$\frac{\partial}{\partial\alpha} [\ell n(L)] = 0$$

$$\frac{\partial}{\partial\beta} \ell n(L) = 0 \tag{14}$$

Differentiating equation (13) partially with respect to α, it follows that

$$\frac{n}{\alpha} + \sum_{i=1}^{n} \ell n(t_i/\beta) - \sum_{i=1}^{k} (\frac{t_i}{\beta})^{\alpha} \ \ell n(\frac{t_i}{\beta}) = 0 \tag{15}$$

Next differentiating with respect to β, it can be shown that

$$-n \frac{\alpha}{\beta} + \frac{\alpha}{\beta} \sum_{i=1}^{k} (\frac{t_i}{\beta})^{\alpha} = 0. \tag{16}$$

If $\frac{\alpha}{\beta} \neq 0$, then equation (16) reduces to

$$\beta = [\frac{1}{n} \sum_{i=1}^{k} (t_i)^{\alpha}]^{1/\alpha}. \tag{17}$$

Substituting equation (17) into (15),

$$\frac{n}{\alpha} + \sum_{i=1}^{n} [\ell n\ (t_i) - \frac{1}{\alpha}\ \ell n\ \{\frac{1}{n} \sum_{i=1}^{k} (t_i)^{\alpha}\}]$$

$$- \sum_{i=1}^{k} (\frac{(t_i)^{\alpha}}{\frac{1}{n} \sum_{i=1}^{k} (t_i)^{\alpha}} [\ell n\ t_i - \frac{1}{\alpha}\ \ell n\ \{\frac{1}{n} \sum_{i=1}^{k} (t_i)^{\alpha}\}]) = 0. \qquad (18)$$

Multiplying throughout by n (18) takes the final form,

$$\frac{1}{\alpha} + \frac{1}{n} \sum_{i=1}^{n} [\ell n\ (t_i) - \frac{1}{\alpha}\ \ell n\ \{\frac{1}{n} \sum_{i=1}^{k} (t_i)^{\alpha}\}]$$

$$- \sum_{i=1}^{k} [\frac{{t_i}^{\alpha}}{\sum_{i=1}^{k} (t_i)^{\alpha}} \{\ell n\ (t_i) - \frac{1}{\alpha}\ \ell n\ [\frac{1}{n} \sum_{i=1}^{k} (t_i)^{\alpha}]\}] = 0 \quad (19)$$

Equation (19) is non-linear in α and has to be solved by numerical means. Newton Raphson iteration technique is employed presently to solve for α. The value of α thus obtained is the maximum likelihood estimator $\hat{\alpha}$ substituting which in equation (17) the corresponding maximum likelihood estimator $\hat{\beta}$ is obtained.

Significance Tests

After the parameters α and β of the proposed Weibull distribution model are estimated, it is desirable to check the goodness-of-fit by a significance test. The purpose of this is to examine at what significance level the proposed model accurately represents the data. Towards this end, the classical Chi-square and Kolmogorov-Smirnov tests are employed. The null hypothesis for the model verification can be written as follows.

H_0: The fatigue failure time T is Weibull with $\alpha = \hat{\alpha}$ and $\beta = \hat{\beta}$

Chi-square test is used at 5% and 1% significance levels. Even though the Kolmogorov-Smirnov test was originally formulated for use with models that are obtained wholly independent of data, the test is often used when the parameters have been estimated from data. The combination of MLE and significance tests is employed for the data on station/station groups formed previously and the results are presented in the subsequent sections.

Runout Data

During the inspection, no failure has been observed in some of the aircraft at some stations. Therefore, no time to failure 't' can be recorded. However, such an observation does contain some information that the time to failure is greater than the flight hours completed prior to the inspection. This information is incorporated into the MLE program.

Analysis of data on two fleets of separate aircraft as described in the preceding sections yields the values of α and β as illustrated in Tables 5 and 6. Table 6 results include the runout data while the results in Table 5 do not incorporate the runout data. The inference from the Chi-square test and also from the Kolmogorov-Smirnov test is that the hypothesized distribution does not accurately represent the observed phenomenon. Then, the alternatives are either to change the hypothesized distribution or to perform censoring so that the simple and well known form of Weibull distribution can be preserved.

Censoring Operation

Censoring means discarding portions of data as belonging to some other population. This can have an upper limit and a lower one. The upper limit is denoted as the high outlier limit while the lower limit is called the low outlier limit. Only data lying in between these two limits is considered now. The values of α and β with different levels of censoring and the results of significance tests are shown in Table 17. The calculations show that an acceptable Weibull distribution for B series fleet can be obtained if the high out-lier limit is taken as 6000 hours, and the low out-lier limit is zero. 'E' series fleet however, needs a low out-lier limit of 2500 hours and high out-lier limit of 5000 hours, which means that data from 272 aircraft out of a fleet of 312.

Table 15. Complete Fleet (excluding runouts)

Series	Population	Alpha	Beta	Chi-Square Test Accepted or Rejected at 5% Significance Level	Kol-Smir Test Accepted or Rejected at 5% Significance Level
B	125	5.812	4715	Rejected	Rejected
E	291	2.645	5483	Rejected	Rejected

Table 16. Complete Fleet

Series	Population	Alpha	Beta	Chi-Square Test Accepted or Rejected at 5% Significance Level	Kol-Smir Test Accepted or Rejected at 5% Sig-Level	Run out data Included or Not
B	130	5.061	4860	Rejected	Rejected	Yes
E	312	2.588	5702	Rejected	Rejected	Yes

Table 17. Fleets with Censored Data

Series	Population	Levels of Censoring	Alpha	Beta	Accepted or Rejected at 5% Sig. Level		Is run out data included or not
					Chi-Square Test	Kol-Smir Test	
B	119	Above 6000	6.2328	4612	Rejected	Rejected	Yes
B	114	Above 6000	7.348	4520	Accepted	Accepted	No
E	155	Above 5000 & Below 2500	6.962	3823	Rejected	Accepted	No

The results for all the stations/station groups with/without censoring is presented in Tables 18 to 21.

Three Parameter Weibull Distributions

In many cases the two parameter Weibull distribution has not been an accurate representation of the data. In particular, no accurate distribution for fleet 'E' is representative of the major portion of the sample size of 312 aircraft. Then, it is desired to explore the possibility of replacing the two parameter Weibull model by a three parameter one. The cumulative distribution function of the three parameter Weibull model is given below.

$$F_T(t) = 1\text{-}\exp[-(\frac{t-t_o}{\beta-t_o})^{\alpha}] \qquad (3\text{-}20)$$

A computer program is written to estimate the three parameters α, β, and t_o by MLE. The results of the MLE and the significance tests are illustrated in Table 22. As can be seen, the three parameter Weibull distribution also does not provide any acceptable models for E-series fleet. So far the only information utilized is the specially defined 'failure time, T' from the data. If the failure criterion is recapitulated, no consideration is given to the length of the fatigue crack. The mere existence of the crack irrespective of its length at an inspection time is noted as the failure time. The following section refers to a method of

Table 18. Base Groups--Complete Fleet

Series	Base	Population	Alpha	Beta	Accepted or Rejected at 5% Significance Level		Is run out data included
					Chi-Square Test	Kol-Smir Test	
B	Clark	84	5.644	4997	Rejected	Rejected	Yes
E	Ching Chaun Kang	91	2.7101	5334	Rejected	Rejected	Yes
E	Sewart	54	3.969	4365	Rejected	Rejected	Yes

Table 19. Station Groups--Complete Fleets

Series	Station Group	Population	Alpha	Beta	Accepted or Rejected at 5% Sig. Level: Chi-Square Test	Accepted or Rejected at 5% Sig. Level: Kol-Smir Test	Is it much better than 5%	Is run out data included
B	3-8	130	2.604	11,074	Rejected	Rejected	--	yes
B	9-10	130	3.476	6,987	Rejected	Rejected	--	yes
B	11-16	130	2.236	11,768	Rejected	Rejected	--	yes
B	17	130	4.666	6,854	Accepted	Accepted	yes	yes
B	18-28	130	2.728	10,436	Rejected	Rejected	--	yes
B	29,30, 59,60	130	4.566	6,844	Accepted	Accepted	yes	yes
B	31,32	130	4.706	6,859	Accepted	Accepted	yes	yes
B	33-38	130	4.281	6,931	Rejected	Rejected	--	yes
B	39-40	130	3.519	7,029	Rejected	Rejected	--	yes
B	41-46	130	4.264	7,010	Accepted	Accepted	yes	yes
B	47	130	4.764	6,901	Accepted	Accepted	yes	yes
B	48-58	130	2.235	11,211	Rejected	Rejected	--	yes
B	61-72	130	4.525	6,918	Rejected	Rejected	--	yes
B	73-76	130	3.304	5,906	Rejected	Rejected	--	yes
B	77-88	130	4.309	6,921	Rejected	Rejected	--	yes
B	89-92	130	3.212	6,006	Rejected	Rejected	--	no

Table 19 (continued)

Series	Station Group	Population	Alpha	Beta	Accepted or Rejected at 5% Sig. Level: Chi-Square Test	Accepted or Rejected at 5% Sig. Level: Kol-Smir Test	Is it much better than 5%	Is run out data included
E	3-8	33	4.747	7,490	Accepted	Accepted	Yes	No
E	9-10	72	3.372	3,835	Rejected	Rejected	--	No
E	11-16	40	4.044	7,094	Rejected	Accepted	Yes (K-S)	No
E	33-38	33	5.00	7,514	Accepted	Accepted	Yes	No
E	39-40	66	3.443	4,021	Rejected	Rejected	--	No
E	41-46	43	4.542	6,886	Accepted	Accepted	Yes	No
E	61-72	40	1.994	5,243	Accepted	Accepted	Yes	No
E	73-76	228	2.622	5,417	Rejected	Rejected	--	No
E	77-88	28	1.979	4,856	Accepted	Accepted	Yes	No
E	89-92	247	2.733	5,665	Rejected	Rejected	--	No

Table 20. Station Groups--With Censoring

Base	Station Group	Level of Censoring	Population	Alpha	Beta	Accepted or Rejected at 5% Sig. Level		Is it much better than 5%	Is runout data included
						Chi-Square Test	Kol-Smir Test		
B	3-8	Above 6,000	29	8.446	4,865	Accepted	Accepted	Yes	No
B	9-10	Above 8,000	80	4.270	5,636	Accepted	Accepted	Yes	No
B	11-16	Above 6,000	31	6.358	4,767	Accepted	Accepted	Yes	No
B	17	Above 7,200	91	6.008	6,045	Accepted	Accepted	Yes	No
B	16-28	Above 6,000	28	9.129	4,676	Rejected	Accepted	Yes(k-s)	No
B	29,30, 59,60	Above 6,500	103	5.848	6,269	Rejected	Accepted	Yes(k-s)	No
B	21-32	Above 6,500	73	6.295	5,705	Accepted	Accepted	Yes	No
B	33-38	Above 6,500	64	6.671	5,448	Accepted	Accepted	Yes	No
B	39-40	Above 6,000	57	5.953	4,895	Accepted	Accepted	Yes	No
B	41-46	Above 8,000	94	5.446	6,235	Accepted	Accepted	Yes	No
B	47	Above 7,500	111	5.939	6,414	Accepted	Accepted	Yes	No

Table 20 (continued)

Base	Station Group	Level of Censoring	Population	Alpha	Beta	Accepted or Rejected at 5% Sig. Level: Chi-Square Test	Accepted or Rejected at 5% Sig. Level: Kol-Smir Test	Is it much better than 5%	Is runout data included
B	48-58	Above 5,500	33	7.841	4,502	Accepted	Accepted	Yes	No
B	61-72	Above 7,000	76	6.248	5,800	Accepted	Accepted	Yes	No
B	73-76	Above 6,000	80	11.641	4,542	Accepted	Accepted	Yes	No
B	77-88	Above 6,000	62	6.155	5,410	Accepted	Accepted	Yes	No
E	3-8	Below 5,000	24	8.304	8,161	Accepted	Accepted	Yes	No
E	9-10	Above 1,500	99	8.805	1,094	Accepted	Accepted	Yes	No
E	11-16	Below 5,000	25	6.888	8,068	Accepted	Accepted	Yes	No
E	33-38	Below 4,500	28	7.121	7,894	Accepted	Accepted	Yes	No
E	39-40	Below 4,000	14	5.655	5,983	Accepted	Accepted	Yes	No
E	41-46	Below 3,500	42	4.730	6,940	Accepted	Accepted	Yes	No
E	61-72	Below 4,500	18	5.559	7,750	Accepted	Accepted	Yes	No
E	73-76	Above 4,300	127	6.510	3,578	Accepted	Accepted	Yes	No
E	77-88	Below 1,500	26	2.203	5,134	Accepted	Accepted	Yes	No
E	89-92	Above 3,900	107	7.812	3,443	Accepted	Accepted	Yes	No

Table 21. Base Groups--With Censoring

Series	Base	Level of Censoring	Population	Alpha	Beta	Accepted or Rejected at 5% Sig. Level: Chi-Square Test	Accepted or Rejected at 5% Sig. Level: Kol-Smir Test	Is it much better than 5%	Is runout data included
B	Clark	Above 6,000	80	9.045	4,738	Accepted	Accepted	Yes	No
E	Ching Chaun Kang	Below 2,000 & Above 5,000	50	10.629	4,147	Accepted	Accepted	Yes	No
E	Sewart	Above 4,400	43	9.62	3,706	Rejected	Accepted	Yes (k-s)	Yes

Table 22. Three Parameter Weibull Model for E-Series

Series	X_o	Population	Shape Parameter α	Scale Parameter β	Accepted or Rejected at 5% Sig. Level x^2-Test	Runout data included or not
E	990	312	2.007	4607	Rejected	Yes
E	1193	310	1.882	4380	Rejected	Yes
E	1331	308	1.770	4216	Rejected	Yes
E	2000	296	1.661	3645	Rejected	Yes
E	2500	293	1.324	3033	Rejected	Yes
E	3000	271	1.091	2597	Rejected	Yes
E	3500	217	1.136	2680	Rejected	Yes

incorporating the crack lengths for fatigue failure data analysis.

Regression Modeling

In this method [102] the failure time corresponding to a fixed value of the crack length is obtained by regression analysis of crack lengths and the corresponding times. Weibull parameters are estimated using these modified times. The estimated parameters are checked for goodness-of-fit by using the Chi-square test. As the fixed crack length for failure criterion is changed, the parameters of the probability distribution of the failure time have changed. A significant inference that can be drawn from this analysis is that the probability distribution for fatigue failure continuously changes as the crack length is increased. This result is indicative of the imminent need for a stochastic process to describe fatigue phenomenon. This is tackled in the coming chapters.

Conclusions

The preliminary analysis of fatigue data yields the following objective conclusions.

(1) it is not feasible to model the entire fleet data, without censoring operations, by a two/three parameter Weibull distribution.

(2) Censoring the high out-liers of 6000 hours an acceptable two parameter Weibull model is obtained for B

series fleet. The value of obtained here is much higher than the value suggested ($\alpha = 4.0$) and used by other investigators.

(3) Base censoring provides an acceptable model only for clark base and with high out-lier censoring of 6000 hours.

(4) With station censoring, five out of ten station groups provide acceptable distributions. The values of β for these cases are higher than the β for the entire fleet. This indicates that the life of the individual structure is longer than the whole aircraft.

(5) Station censoring together with high out-lier censoring provide acceptable distributions for all the cases.

(6) Regression analysis of crack length and time result in increasing values for increased crack lengths.

In summary, the following subjective conclusions are drawn from the preliminary analysis. The fatigue behavior of individual structure differs from that of the whole structure. The relative susceptibilities to fatigue cracking of various parts of the whole structure are brought out. Some of the parts have to be inspected more often than others. It also provides a basic separation of data belonging to separate populations. The probability distribution of failure time changes as the crack length changes. This demands a stochastic process for the proper representation of crack propagation in time.

APPENDIX II

LITERATURE SURVEY

The primary field of the present research is called "structural reliability." The specific problem of interest is concerned with the application of the principles of structural reliability to fatigue design of aircraft structures. Then, the logical assertion is to survey the existing literature in the following secondary fields comprising the primary field of the research problem at hand.

(1) Applied probability, statistics and stochastic models

(2) Deterministic and statistical methods in fatigue

(3) Factor of safety and reliability, and

(4) Reliability based fatigue design.

The following is a concise presentation of the review of literature in the above fields.

The book by Papoulis [1] is by far a classical text on the theory of probability. It tends slightly more towards a theoretical development than application. The book starts with the basic probability theory, then goes into the characterization of random variables and the various distributions and finally tackles the stochastic processes. The fundamentals of statistics and reliability are reviewed by

Amstadter [2]. Logic diagrams, series and parallel components and effect on reliability are discussed. Mathematical modelling and prediction of reliability for components are presented. The apportionment of system reliability to components is discussed. The reliability variation of a system with time is explained.

Benjamin and Cornell [3] have written a book purely from application point of view. It is a fundamental text on probability, statistics and decision from generally a civil engineering point of view. It comprises data reduction, probability theory, probabilistic models, model verification and decision theory. One of the excellent features of this book is the large number of solved practical example problems. Hines and Montgomery's [4] book is thorough and presented in easy to understand fashion. It consists of set theory, random variables, probability theory and statistical methods. The theoretical principles are fortified by good numerical examples.

Heller [5] has reviewed several frequently used probabilistic models and their representation of component life. It is shown that Weibull distribution provides a convenient representation of life type phenomena. Graphical means of determining model parameters is discussed and illustrated with a fatigue example. The book of Tribus [6] is written from a fundamental point of view and emphasizes a particular philosophy of probability and decision making.

Probability is regarded as a numerical encoding of a state of knowledge rather than as relative frequencies or as a ratio of chances among equally likely outcomes. Whether a decision is right or wrong is not to be decided on the basis of whether it turned out to be right or wrong but on the basis of its rationality. The book consists of probability principles, distributions, concepts of expectation and variance, Bayes theorem, prior probabilities and construction of rational descriptions.

A review of statistical methods applicable to structural reliability is presented by Lemon [7]. The concepts of theoretical models, distributions and parameter estimation are explained. Point and interval estimation of parameters is explained. Chi-square and Kolmogorov-Smirnov goodness-of-fit tests are discussed. A fairly thorough treatment of the Weibull distribution is presented. Locks [8] discusses the elementary techniques for assessing the reliability, maintainability or availability of a component or system. It is assumed that the component failure or success data which are used for reliability estimation are governed by some parametric probability distribution. The usual meaning of the term "reliability" is "the probability of performing successfully." The author suggests a strategy for parametric reliability estimation. It consists of identifying the family to which the probability distribution belongs. Then, the "best estimates" of the parameters are determined from test data.

Then reliability is obtained by the appropriate cumulative probability. The confidence level is the probability that the reliability in general is at least the estimated reliability. This is obtained from the entire data. The author defines maintainability and availability for a repairable system. Maintainability is a measure of the effectiveness of performance only during restoration to service. It is characterized by only one random variable, i.e. repair time. On the other hand availability is a measure of the total performance effectiveness. It is characterized by two random variables, i.e. failure time and repair time. The author points out the suitability of Weibull distribution for time to failure data.

Cohen [9] has given a method of determining the parameters of the two parameter Weibull distribution by maximum likelihood estimation. Separate expressions are given for complete samples, singly censored samples and progressively censored samples. An iterative procedure is required to solve the non-linear equations resulting from the Maximum Likelihood Function. In his paper Harter [10] discusses a similar procedure of using the maximum likelihood estimation for the parameters of Gamma and Weibull populations from complete and censored samples. He has pointed out that a location parameter can not always be found to satisfy the three non-linear equations that are developed in his paper. Freudenthal [11] has given an ultimate strength analysis of a large number of data points of different types of aircraft structures. A Weibull distribution is found to be a good representation of ultimate strength.

Whittaker and Besuner [12] have presented a method for specifying a desired reliability for an arbitrary fleet member in the fatigue failure mode. A two parameter Weibull model is employed with the shape parameter being determined from available data. The scale parameter has to be estimated from full scale fatigue tests. The lower interval estimate is proposed to represent the safe life. It is recommended that the scatter factor concept be replaced by theirs. They also have used the maximum likelihood estimate for the parameter. In reference [13] the authors proposed a method for reliability analysis of statically indeterminate structures. Their primary contribution consists of a systematic counting of all the failure modes to arrive at the reliability.

The editors [14] have presented in systematic arrangement a collection of papers on reliability and fault tree analysis (FTA). The usefulness of fault tree analysis in determining the system reliability is stressed. It is contended that FTA is a detailed deductive analysis that usually requires considerable system information, and as such can be a useful design tool and/or diagnostic tool. It predicts the most likely causes of system failure in the event of a system breakdown. The goal of the fault tree is to model the system conditions that can result in the undesired event. It represents graphically as well as logically the various combinations of possible events--either normal or abnormal occurring in a system. This systematic break-down of events aids the system reliability estimation.

Carter [15] has written a concise but clear to understand book on mechanical reliability from the point of view of the mechanical engineer. Primarily the book is concerned with what reliability is and then how to achieve high reliability for mechanical equipment. From among a large number of available definitions of 'reliability' Carter has cited the following three essential features

(1) a quality of performance is expected,

(2) this is expected over a period of time, and

(3) reliability is expressed as a statistical probability.

Accomplishing high reliability means elimination of failures to the extent possible. Failures are due to either bad design, bad manufacture, bad operation or bad luck. In order to build a complete appraisal of these factors involved, Carter has suggested a "circular" approach involving design-manufacture-operation-failure analysis-redesign. Hence, Chapters 3, 4, 5, and 6 of his book are devoted to design, manufacture, operation and chance, respectively. John Bompas-Smith [16] has written a book on "Mechanical Survival: the use of reliability data." The initial chapters are allocated to putting values to reliability, mortality curves and failure rates. Next, a theory of failures is presented where failure is ascribed to 'duty' and strength interaction. Normal, Weibull, Binomial and Poisson distributions for 'duty' and strength are considered.

The Agardograph edited by Liebowitz [17] can be considered to be a classical exposition of fracture mechanics as an interdisciplinary science. The chapters are organized so that the reader may obtain an understanding of fracture mechanics concepts and their relationship and application to the unique problem of aircraft design. The importance of fail-safe design concepts in aircraft structural design is strongly emphasized with detailed discussions of the basic concepts and their applications to design, materials and testing. Many new approaches and tools have been presented to predict crack growth and critical crack lengths. Descriptions are included on improved non-destructive testing methods and the use of acoustic emissions, surface dye penetrants, magnetic particle testing holography and fractography to facilitate the determination of flaw sizes and cracks. There is a chapter which treats the present reliability of crack detection methods and the means for determining the crack size. Also, the appendices contain detailed information on typical plane strain fracture toughness of aircraft materials, fracture toughness test results for the same materials, and the titles and references of about 140 configurations for which stress intensity factors have been determined.

Brock [18] presented the basic information on fail-safety and fail-safe design concepts and fatigue crack propagation in aircraft structures. Fail-safety is defined as the capability of the structure to sustain an appreciable load under the presence of cracks or failed parts. It also requires that the damage can be detected before it has extended to calamitic proportions. The various methods of attaining fail-safety that are proposed are

(1) Provision of multiple load path

(2) Material selection and proper design with particular regard to crack propagation rate

(3) Performing periodic proof loading, and

(4) Performing periodic stripping.

Brock [19] discussed fail-safe design concepts with reference to fatigue crack propagation in stiffened panels. In order to apply the fail-safe concept it is necessary to make reliable estimate of the number of flight hours that elapse between crack initiation and the reaching of critical size. Inspection intervals will have to be based on this estimate such that the crack propagation takes two or three inspection periods. The prediction of fatigue crack propagation rates and crack propagation time should be made on the basis of relevant data for fatigue loads, crack propagation data and structural geometry.

Mechanics of Metals by D'Isa [20] deals comprehensively with elasticity, theories of failure plasticity, creep and fatigue. The text is designed to meet the needs and interests of aeronautical, civil, material-science, mechanical metallurgical and structural engineers. Not only theoretical methodologies are presented in a thoroughly understandable manner but numerous examples are worked out to illustrate these methodologies. The chapter on fatigue is very enlightening with information on the nature and mechanisms of fatigue phenomenon.

Hardrath [21] discusses the principles of fatigue and fracture mechanics. The discussions include S-N curves, mean and alternating stresses, stress concentration, cumulative damage and crack propagation. Forman, Kearney and Engle [22] propose a modified crack growth law which includes the effect of mean stress. It also satisfies the condition of infinite growth rate as the stress intensity factor approaches the critical value (fracture toughness of the material). Poe [23,24] has dealt with the problem of fatigue crack propagation in stiffened panels. Superposition principle has been utilized to advantage in solving for the stress intensity factor for the crack in stiffened panel. The effect of the stiffeners (stringers) is to reduce the stress of the tips of the crack which is implied in the "tip stress reduction factor." The numerical results for this function are presented for various geometrical configurations. Flieger and Brock [25] have analyzed the residual strength variation in cracked stiffened sheet structures, considering a state of plane-stress. They have related the residual strength of stiffened panel to that of unstiffened panel by means of two factors C_R (tip stress reduction factor) and L_S (stringer load concentration factor). The effect of the crack is considered through the stress intensity factor.

Heller and Donat [26] have considered the reliability of a multiple load path structure and analyzed interns of a 'defined' risk function. Fatigue damage is assumed to accumulate linearly until the remainder of the cross section is broken by a single application of the next load cycle. Lambert and Troughton [27] discuss the fail-safe philosophies in aircraft structural design. A brief account of the advantages and disadvantages of fail-safe method is given. The cited merits are improved safety, weight savings, full availability of the potential fatigue life of each individual aircraft of the fleet and protection against accidental damage in service and manufacturing errors. The only disadvantage is the possible continued heavy in-service inspection time required. Swift and Wang [28,29] have shed more light on the damage tolerant design methodology of aircraft structures from a deterministic point of view. They have included the test verification of fuselage structure and wing structure of DC-10 aircraft.

The use of statistical methods in fatigue data analysis is described by Wirsching and Yao [30]. They have contended that Weibull distribution is appropriate for representing fatigue life at a given stress level. Properties of Weibull distribution are discussed and methods of estimating Weibull distribution parameters and confidence limits are presented.

Reference [31] is a systematic compilation of articles edited by Madayag encompassing deterministic as well as statistical aspects of fatigue. This is an excellent coverage from a view point not only of theory but also of practical design considerations. The various cumulative damage theories are explained at length, in addition to the basic concepts related to the fatigue process in the first chapters. The latter part of the book focuses on fatigue analysis with reference to random loading measurement and fatigue design. Application of the fatigue design methodology is demonstrated in the last chapter for a helicopter design and service problem. The extent of the statistical fatigue design methodology is restricted in the sense that no stochastic modelling for fatigue phenomenon is considered.

Butler [32] has investigated the application of reliability analysis methods to the estimation of probable fatigue performance. Order statistics theory is used to define the reliability of a fleet of aircraft or a number of fatigue exposed details. A reliability analysis plan is presented and compared with the previous method of fixed-scatter factor in the fatigue crack-free service periods. Both the two parameter Weibull distribution and Log normal distribution with empirically defined shape parameters are employed to investigate the application of order statistics. Maximum likelihood estimators considering only the first two ordered fatigue failures are described and used to assess fatigue data and to establish distribution shape parameter values.

Among the conclusions drawn, the following are specific to the current research at hand.

1. The two parameter Weibull distribution shape parameter is estimated to be 4.0 for aluminum alloys.

2. The two parameter Weibull distribution provides a lower safe life estimate than log-normal distribution.

3. The two parameter Weibull distribution is more sensitive to high levels of reliability than log-normal distribution.

4. The two parameter Weibull distribution is less sensitive to the fatigue test sample size.

5. The Weibull distribution has a failure rate which increases monotonically and rapidly while log-normal distribution failure rate increases initially but eventually decreases to zero.

Weiss and Baker [33] have elaborated on the causes of "product liability" which is emerging as a new facet of "reliability based product design." The authors have pointed out emphatically that the increasing propensity of material failures due to higher strength levels of new structural materials and thinner sections have given rise to higher probability of sudden catastrophic failure. It has been suggested that designs should be oriented towards a finite life time, in other words towards a known risk. The material failure modes are characterized by

a. fatigue failures

b. corrosion and stress corrosion failures

c. overload, and

d. brittle fracture.

The author's conclusion is that the design must take into account all the difficulties inherent to the product. This necessitates an increasing participation of all contributors to new design methodologies such as reliability based design.

Yang and Heer [34] have proposed an approach for calculating reliability based on a residual strength concept. Residual strength is determined by means of fracture mechanics principles. Then they have derived an expression for the residual strength R_n after n cycles as a function of the load spectrum. From this, the rick function and reliability are inferred.

Karnopp and Valls [35] have discussed their experience in using a finite element stress program to predict component reliability. It is pointed out that the techniques for achieving a reliable mechanical design have progressed well beyond the conventional factor of safety approach. Simple analytical formulas usually suffice to relate the probabilistic quantities to stresses or other failure criteria. For complex structural configurations numerical methods of stress evaluation have to be employed, usually in a digital computer. Because of their complexity they were not set up for use in reliability calculations. Motivated by this the authors have devised a partial derivative method using finite approximations to the derivatives for adapting complex computer programs for use in probabilistic structural analysis.

Freudenthal [36] has discussed in one of his earlier papers the statistical aspects of fatigue. He has indicated the desirability of statistical models for fatigue data in order to derive better interpretations from the test data. A log-normal distribution has been derived for the fatigue life cycles under constant amplitude stress. References [37-39] are excellent reviews of the statistical analysis of fatigue data. They have described basics of statistical analyses and methods used to interpret and convert these analyses into useful engineering information. Smith [40] has investigated the crack growth under random loading by employing power spectral methods in the experimentation. McMillan and Pilloux [41] have determined the influence of the maximum stress, the range of stress and the sequence of loading on the rate and mechanism of crack propagation in 2024-73 aluminum alloy by means of electron fractography. Paris' equation is used for both programmed and random loading cases to calculate the expected rate of growth of crack.

Bastenaire [42] has described the probabilistic description of constant stress amplitude fatigue test results. This description includes S-N curves, P-S curves, P-N curves and accounts for the occurrence of runouts. The method of estimation of model coefficients consists of the weighted least squares method. Statistical distribution of number of cycles to failure ϕ(NCF) is derived assuming S and N as independent random variables. Mukherjee and Burns [43] have employed regression models based on linear elastic fracture mechanics to interpret the effect of stress ratio on fatigue crack growth rate in 7075-T6 aluminum alloy. It is shown that fatigue crack growth rate can be related to range and maximum values of stress intensity factor. That the effect of compression cycle is to aggravate the crack propagation is claimed by the authors.

Feddersen [44] has discussed the evaluation and prediction of residual strength of centercracked panels. The points are made that (a) stress intensity factor can be utilized effectively in an elementary format to generate smooth and continuous stress-flow size relation, (b) stress intensity factor is an accurate and consistent measure of fatigue damage and (c) panel width can be uncoupled as an independent parameter in crack behavior.

Taub [45] has put forward the philosophical outline of the probability of failure with respect to the structural safety of aircraft. It has been recognized that the problem of structural safety comprises five more or less distinct problem areas as follows.

Problems associated with (i) the choice and definition of applied loads, (ii) analysis of internal loads, (iii) calculation of stress distribution due to internal loads, (iv) local and overall stability of structure, and (v) static and fatigue strength of structure or material.

Brussert [46] has expressed the necessity for an approach to predict crack growth to failure. The approach would treat crack growth as a continuous stochastic process. Each different crack description and length would be a state variable in this process with residual strength failure constituting the terminal state. By present fracture mechanics theory a residual strength failure occurs when the imposed maximum stress intensity exceeds the material fracture toughness.

References [47-55] deal with design factors and their relation to reliability or probability of failure. The usefulness of probabilistic approaches to the design of machine members is pointed out by Kececioglu and Haugen [47], Haugen [48], Bury [51] and Disney and Charles [53]. They have presented various methods to relate factor of safety (defined as a random variable) to reliability, such as stress-strength interference method when the density functions of both are given. If the densities are not known approximate statistical means have been suggested by Mischke [50]. My Dao-Thien and

Massond [55] have derived an improved version of the relationship of deterministic factor of safety to reliability. The derivation does not require the knowledge of the probability distributions of stress and strength. Only the statistical measures (mean and variance) are sufficient. They have also presented design nomograms for practical use. References [49] and [54] discuss the procedures to develop probability density functions for stress and strength from available data on them. Disney and Charles [53] and Disney and Sheth [54] have manipulated formulas for calculating the probability of failure for several commonly used distributions on stress and strength into suitable forms for computation. Some are closed form solutions and others require numerical methods.

It is suggested by Bouton, Trent and Chenoweth [56] that the design factor be a variable which is selected for each particular structure and is dependent on the use of the structure and quality of the structure. Uses are classified as low risk, standard risk and high risk with specific reliability goals for each. The structural quality is represented by the strength coefficient of variation.

Ang and Amin [57] have introduced the concept of "probability of unsafety." The factor of uncertainty is to be applied after all available probabilistic information has been incorporated into the stress and strength. This factor is to account for all undeterminable errors and inadequacies. Bouton and Trent [58] have given a detailed presentation of a new proposed method of establishing and using structural design factors is given. The safety factor is considered as a variable quantity which is selected for each separate design. The value selected depends on the desired reliability and on the strength scatter for the structure. A structural code is suggested by Cornell [59] in which the safety factor selected depends only upon the means and coefficients of variations of stress and strength but not on the specific probability distributions. However the method is arbitrary in the sense a safety index has to be selected by the designer.

Freudenthal [60] has investigated safety, reliability with reference to the design of structural members. The safety factor is treated as a statistical distribution function of the ratio of stress and strength. Reliability is expressed as a function of the life of the structure in terms of the number of load cycles. In this form, the effects of fatigue and ultimate load failures can be combined by adding the respective probabilities.

In a report [61] the authors have defined factor of safety as a ratio of strength to stress. Here both the strength and stress are treated as random variables and so is the factor of safety. The probability of failure is the probability of the random variable safety factor being less than or equal to unity. Ghare [62] has recognized two factors as affecting the strength of components and thereby the reliability. They are (i) the safety factor which determines the mean resisting strength and (ii) the quality factor associated with production which determines the variability of the strength.

Svenson [63] has discussed the mathematics of determining probability of failure and required mean factor of safety when several random variables contribute to the variations in stress and strength. Design charts are given to determine safety factor. Sensitivity of the safety factor to coefficient of variation of stress and strength is given.

Davidson [64] has emphasized the necessity of inspection scheduling for the sake of optimum reliability. A method has been presented to quantify the effect of inspection on reliability.

Moon [65] has computed "design allowables" for primary strength properties and typical values are quoted for secondary properties such as elastic moduli, fatigue, creep and fracture toughness. These are design allowables established on a probability basis.

The behavior of pin ended column subjected to random loading is studied by Wirsching and Yao [66]. This is done by using an analog simulation in which frequency and magnitude of load are limited. The applied load is divided into a static load and a random load. The power spectral density of the random load which causes instability is measured for several values of static load.

The reliability of redundant structures from the fail-safe standpoint is considered by Shinozuka and Itagaki [67]. They have shown that the probability that a brittle structure could survive after failure of one member is very small. Whereas for ductile structures the same probability is higher.

A statistical model for the fatigue process is used by Payne [68] to carry out the reliability analysis. The statistical variability in growth rate and residual strength is included together with the inspection procedure.

Lalli and Kececioglu [69] have given a methodology for determining the reliability of structures. The reliability methodology reduces the number of 'modifying factors' and thereby results in a lighter design. The following information is required for the proposed methodology:

(a) statistical strength distribution for material,

(b) statistical distribution of loading, and

(c) statistical methods for relating stress to strength.

Ang [70] has described a fundamental extension of the classical reliability design concept. This generalization is necessary to allow the consideration of subjective as well as objective factors in engineering design. Objective uncertainties are treated in explicit probability terms whereas subjective uncertainties are handled through a "judgement factor." The risk of failure is then the product of the two.

Freudenthal [71] has proposed an approach to structural reliability based on order statistics and the expected time to first failure in a given fleet. A modified fatigue sensitivity factor is developed for the correlation of ultimate load and fatigue design. He has justified the Poisson process representation for the step like, zig-zag propagation of fatigue crack length. Graziano and Fitch [72] have conducted full scale fatigue tests on an aircraft wing and service connected fatigue failure data are collected and compared with test results. Miner's rule of cumulative fatigue damage is used to predict the lives to initiate fatigue cracks. Paris and Forman growth expressions are employed.

Eggwertz [73] has investigated fatigue life and residual strength of a wing panel for reliability purposes. In the reliability analysis of a fail-safe structure, statistical information regarding service time until crack initiation as well as subsequent reduction in residual strength is indispensable. It is shown that the relationship between residual strength and critical crack length shows little stochastic variation. An expected number of up-crossings of the residual strength by the load is employed for probability of failure. Smith [74] has evaluated experimentally the random loading fatigue crack growth behavior of some aluminum and titanium sheet materials. Narrow-band and broad-band random loading crack growth rate behavior is investigated by means of power spectral methods in the experimentation and random load analysis.

Shinozuka and Yang [75] have proposed an approach in structural optimization based on reliability analysis with an emphasis on the use of proof load testing. Methods of optimizing the weight subject to the constraint on the expected cost are presented. The total expected cost consists of the cost of proof testing and the cost of expected failure.

Structural reliability as a probabilistic phenomenon is discussed by Bouton [76]. Probability of survival is considered to be the quantitative measure of reliability with weight and economics recognized. The effect of variability in loads and strength on reliability is discussed. Bouton and Trent [77] have presented a concept for characterizing fatigue strength as the remaining static strength of a fatigue damaged structure. The basis of the concept is that fatigue failures are in reality ultimate load failures of a fatigue damaged structure. Fatigue failure is hypothesized to occur only when the residual strength is exceeded by an applied load.

The difference between functional and statistical relationships in design is discussed by Freudenthal [78]. Superposition of influences, types of loads (dead or live), temperature effects, wind forces and material properties are also discussed. The author has made an appeal for the consistency of designing for probability of failure rather than for arbitrary design factors [79]. The paper is in the nature of an explanatory exposition on probability and safety factors as applied to structures.

Instead of using a random variable to characterize the loading on a structure, Leve [80] uses a set of deterministic life histories. In this concept, reliability of an element over an individual life history is the minimum value attained along the history. Then, the total reliability is $\sum_i \rho_i R_i$ where ρ_i is the probability of the i^{th} history and R_i is the reliability of the structure for the i^{th} history.

Turkastra's [81] discussion of the failure modes is mostly philosophical and hinges on the minimization of expected loss from choosing among several design schemes. Only a bound on failure probability must be established so that explicit distribution curves are not required. Any design which has a probability of failure less than a specified quantity is assumed to have a zero probability of failure. It is stated that a good approximation to an optimum design is the least costly design which has a probability of failure less than the specified quantity.

Switsky [82,85] has proposed a procedure for designing for minimum weight with given reliability. Techniques for determining the minimum member weights, probability of failure and safety factors are presented. A plot for determining the safety factor for given failure probability is given. Shinozuka [83] has presented a simple model of optimum design based on the expected cost of failure for proof tested components. The cost model includes the cost of proof testing and the expected cost of service failure. The method is shown for statically determinate as well as indeterminate structures. Moses and Stevenson [84] have proposed some methods for incorporating reliability analysis into optimum design procedures by designing for a specified probability of failure. A method of optimum design is presented in the sense that some objective function which is dependent on the design variables is minimized subject to a constraint on reliability. Mau and Sexsmith [86] have investigated structural optimization of expected cost. This is in contrast to previous investigations which minimized cost or weight subject to constraints on the probability of failure. Ghista [87] has proposed a concept for optimizing a structural design with a specified maximum probability of failure. The development is carried out for a two member structure under two independent loading conditions. A trial and error method is suggested for searching for the minimum weight which does not violate the probability of failure constraint.

The relationship of desired structural reliability to such factors as economics and public interest is discussed by Freudenthal [88]. It is recognized that reliability should not be specified arbitrarily but should depend upon the consequences of failure. A model for optimizing the probability of failure based on minimizing the total expected loss is suggested. Heer and Yang [89] have used the proof load level as a design variable and the total weight is optimized with total expected cost as a constraint. The strength distribution of the structure is derived as a Weibull distribution and it is shown how this distribution is truncated for a proof tested structure.

Milton and Feigan have investigated the proportioning of probabilities of failure among structural components in terms of a pre-assigned probability of failure for the entire structure such that the weight is minimum. Elements are dissimilar in loading and resistance. It is shown that heavier parts should be assigned relatively higher probabilities of failure to achieve minimum weight. References [91-94] are additional applications of the optimum reliability design procedures and the principle behind is essentially the same as before.

VITA

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He has coauthored technical publications in the field of structural reliability in well-known journals as well as national conference proceedings. He is a member of the American Society of Mechanical Engineers and the Georgia Tech National Alumni Association.

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